AN APPLICATION OF HYPERDIFFERENTIAL OPERATORS TO HOLOMORPHIC QUANTIZATION

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ABSTRACT. We use a hyperdifferential operator approach to study holomorphic quantization. We explicitly construct the Hilbert space operator which corresponds to a given holomorphic function. We further construct the adjoint and products of such operators and we discuss some special cases of selfadjointness.

INTRODUCTION

A lot of people seem to have been interested in the Weyl–Wigner formalism for quantization in recent years [1–9], and some nice physical results have been obtained. One can use this formalism to define quantal operators corresponding to classical functions on phase space as quadratic forms [3, 5], to study the probabilistic interpretation of quantum mechanics (e.g. 'fuzzy spaces' [4, 6]) or to interpret quantum mechanics as a deformation of classical mechanics (see, for instance, [8]).

In [10], Babbitt introduces a holomorphic quantization procedure (a 'holomorphic' analog to Weyl-quantization). For a certain class of holomorphic functions, he defines corresponding quantal operators, which can be considered as unbounded operators on the Bargmann–Fock–Segal (B.F.S.) Hilbert space [11]. For a justification of the choice of holomorphic functions as dynamical variables and for the relation of the B.F.S.-realization with the usual Schrödinger realization, we refer to [10]. While elaborating his formalism of holomorphic quantization, Babbitt remarks that a connection might exist with hyperdifferential operators. In the paper [12] he refers to, Miller and Steinberg give a survey of the properties of hyperdifferential operators, and apply some of these to Weyl-quantization. They work, however, only with operators on a locally convex space which is much larger than the Hilbert space one considers usually.

In this paper, we make a connection between these two approaches: we start from Babbitt's quantization procedure, we apply hyperdifferential operators, and we restrict to the Hilbert space. As a result of this, we can find rather easily the twisted product of two functions (for the definition of a twisted product with respect to a given quantization map, see, for instance, §2 in [8b]), and we have some technical facilities to ascertain whether certain operators are selfadjoint or not. We shall not follow Babbitt's notations, but stay in the Bargmann Hilbert space [11a] without generalizing any further. Using the formalism developed in [5] or [10], such a generalization would be

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straightforward, but it would make our notations much heavier. For the same reasons of simplicity, we shall restrict our discussion to one-dimensional systems (i.e. a two-dimensional real phase space or a one-dimensional complex phase space), although everything said here could be almost trivially extended to the general case of an n-dimensional phase space.

In Section 1, we introduce the definitions and mention some known results about hyperdifferential operators. In Section 2.1, we show how these hyperdifferential operators can be applied to Babbitt's holomorphic quantization in a formal way (this was done for Weyl-quantization in [12]), and in Section 2.2, we see how these hyperdifferential operators on the big locally convex space can be restricted to densely defined operators on the Hilbert space. In Section 3.1, we construct holomorphic twisted products; in Section 3.2, we prove some results about adjoints, and we give some examples of verifications of selfadjointness. Finally, the limitations of our approach as well as those of Babbitt's are discussed.

1. DEFINITIONS AND SOME BASIC PROPERTIES

The Bargmann Hilbert space is defined as

$$\mathcal{H} = \{f; f \text{ is entire } \mathbb{C} \to \mathbb{C} \text{ and } \int d(\operatorname{Re} z)\, d(\operatorname{Im} z)\, |f(z)|^2 \exp(-|z|^2) < \infty\}.$$

It is, up to unitary transformations, the Hilbert space used by Babbitt in his B.F.S.-realization [10] and by Grossmann [5]. The in product is given by (we denote the measure $d(\operatorname{Re} z)\, d(\operatorname{Im} z)$ by dz):

$$(f, g) = \frac{1}{2\pi} \int dz\, \overline{f(z)}\, g(z) \exp(-|z|^2). \tag{1.1}$$

With the power-series expansions $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$, $g(z) = \Sigma_{n=0}^{\infty} b_n z^n$, this can be rewritten as

$$(f, g) = \sum_{n=0}^{\infty} \overline{a_n} b_n n! \qquad \text{(see [11a])}. \tag{1.2}$$

With this in product, $\mathcal{H}$ is a Hilbert space. We denote the topology on $\mathcal{H}$ defined by its Hilbert norm by τ_2. On $\mathcal{H}$, we can define unbounded creation and annihilation operators. Define

$$D(A^*) = \{f \in \mathcal{H}; \int dz\, |z f(z)|^2 \exp(-|z|^2) < \infty\},$$

$$D(A) = \{f \in \mathcal{H}; \int dz\, |(d/dz) f(z)|^2 \exp(-|z|^2) < \infty\}.$$

These linear subspaces are dense in $\mathcal{H}$ (they contain the polynomials). The creation, resp. annihilation operators A^*, A are then defined by:

$$A^*: D(A^*) \to \mathcal{H}, \quad f \mapsto zf; \qquad A: D(A) \to \mathcal{H}, \quad f \mapsto (\mathrm{d}f/\mathrm{d}z).$$

In his holomorphic quantization [10], Babbitt considers the class of entire functions of exponential type in two variables:

$$\mathrm{Exp}_2 = \{F \text{ entire } \mathbb{C}^2 \to \mathbb{C};\ \exists M_F, K_F \text{ such that } |F(z_1, z_2)| \leqslant M_F \exp(K_F(|z_1| + |z_2|))\}.$$

For any F in this class, $F(z, w) = \Sigma_{n,m}\, a_{nm} z^n w^m$, Babbitt considers a corresponding formal operator $QF = \Sigma_{n,m}\, b_{nm} A^{*n} A^m$, where the b_{nm} are computed in function of the a_{nm} (see [10]). Since $\Sigma_{n,m}\, b_{nm} z^n (\mathrm{d}/\mathrm{d}z)^m$ is the usual form of a hyperdifferential operator, it is natural to try to introduce them at this point.

Hyperdifferential operators are defined on the holomorphic function space $E = \{f \text{ entire } \mathbb{C} \to \mathbb{C}\}$. On E we define a locally convex topology by the set of norms $\|f\|_R = \sup_{|z|=R} |f(z)|$ with $R > 0$. This Fréchet-topology is exactly the topology of uniform convergence on compact sets; we denote it by τ_E. For any linear operator A on E, continuous with respect to τ_E, we define its Fourier–Borel symbol by $\hat{\sigma}A(z, \xi) = \exp(-\xi z) A \exp(\xi z)$, where ξ is considered a complex parameter. A continuous linear operator A is called a hyperdifferential operator iff $\exists a_{mn}$ such that $A = \Sigma_{m,n=0}^{\infty}\, a_{mn} z^m (\mathrm{d}/\mathrm{d}z)^n$. One can then prove the following theorems (see [12] and the references mentioned there).

THEOREM 1.1. *Let A be a bounded operator on E, and $\hat{\sigma}A$ its Fourier–Borel symbol. Then $\hat{\sigma}A$ is an entire function in two complex variables and $A = 0$ is equivalent with $\hat{\sigma}A = 0$.*

THEOREM 1.2. *Let A be a bounded operator on E, and $\hat{\sigma}A$ its Fourier–Borel symbol. Then the following properties are equivalent:*

(1) *A is a hyperdifferential operator*;

(2) $\forall R > 0$, $\exists S$, C_R *such that* $\forall \xi$: $\sup_{|z|=R} |\hat{\sigma}A(z, \xi)| \leqslant C_R \exp(S|\xi|)$;

(3) $\hat{\sigma}A(z, \xi) = \Sigma_{m,n}\, a_{mn} z^m \xi^n$ *with* $\forall R > 0$, $\exists S$, C_R *such that* $\forall m, n$: $|a_{mn}| \leqslant C_R R^{-m} S^n (n!)^{-1}$.

Moreover, to any entire function $\hat{\sigma}$ on $\mathbb{C}^2$ satisfying either (2) *or* (3), *there corresponds a hyperdifferential operator A with symbol $\hat{\sigma}A \equiv \hat{\sigma}$.*

Using both Theorems 1.1 and 1.2, it is easy to see that any bounded linear operator on E must be hyperdifferential (the boundedness of A forces $\hat{\sigma}A$ to satisfy Condition (2) in Theorem (1.2), and that any hyperdifferential operator is completely and uniquely determined by its Fourier–Borel symbol. It is possible to define the Fourier–Borel symbol of a product of operators directly from their respective Fourier–Borel symbols:

THEOREM 1.3 (see [12]): *Let A, B be hyperdifferential operators on E, then AB is a hyperdifferential operator on E and*

$$\hat{\sigma}(AB)(z, \xi) = \exp(\partial_z \partial_\xi)(\hat{\sigma}A(z_1, \xi)\, \hat{\sigma}B(z, \xi_1))|_{z_1 = z,\ \xi_1 = \xi} = \sum_{n=0}^{\infty} \frac{1}{n!}\, \partial_\xi^n \hat{\sigma}A(z, \xi)\, \partial_z^n \hat{\sigma}B(z, \xi).$$

2.1. FORMAL DEFINITION

In this section, we sketch the formal definition of the operators QF given by Babbitt in [10]. We then compute the Fourier–Borel symbol of such an operator, and show that this symbol satisfies Condition (2) in Theorem 1.2. Hence, it defines a hyperdifferential operator $\tilde{Q}F$, which we shall take as our starting point for the following.

In the usual Weyl-quantization, the symplectic Fourier transform of a given classical function f is used to define formally its quantal counterpart Qf [3, 14]. The same is done here; however, since we are considering a different kind of function space, a different transformation will be used. Let E_2 be the set of entire functions in two complex variables, equip this vector space with the topology of uniform convergence on compact sets and define E'_2 to be its dual, i.e. the space of continuous linear functions on E_2 with respect to this topology. The set of linear functionals $\{\delta^{(m,n)}; m, n \in \mathbb{N}\}$ with

$$\delta^{(m,n)}(F) = (-1)^m \, [(\partial_{z_1})^m (\partial_{z_2})^n F(z_1, z_2)_{z_1 = z_2 = 0}$$

is obviously a subset of E'_2. One can now prove the following theorem (the proof is an obvious generalization of the proof of Theorem 1 in [12]).

THEOREM 2.1. *Let F be an element of* Exp_2 : $F(z_1, z_2) = \Sigma_{m,n} (m!n!)^{-1} F_{mn} z_1^m z_2^n$. *Then* $\nu_F = \Sigma_{m,n} (m!n!)^{-1} F_{mn} \, \delta^{(n,m)}$ *is an element of* E'_2. *The map* $F \to \nu_F$ *is an isomorphism mapping* Exp_2 *onto* E'_2. *Its inverse is given by* $E'_2 \to \mathrm{Exp}_2 : \nu \to F_\nu$, *where* $F_\nu(z_1, z_2) = \nu(\phi_{z_1,z_2})$ *with* $\phi_{z_1,z_2}(u_1, u_2) = \exp(z_1 u_2 - z_2 u_1)$. (2.1)

This isomorphic correspondence, called the Fourier–Borel transform, will play here the role the symplectic Fourier transform played in the Weyl-quantization.

One usually summarizes Theorem 2.1 by writing Correspondence (2.1) as

$$F(z, w) = \int \exp(zw' - wz') \, \mathrm{d}\nu_F(z', w'). \qquad (2.1')$$

For any function F in Exp_2, Babbitt defines (formally) the corresponding operator by

$$\tilde{Q}F(A^*, A) = \int \exp(A^* w' - Az') \, \mathrm{d}\nu_F(z', w') \qquad (2.2)$$

where A^* is the multiplication operator with z, and A is the derivation operator $\mathrm{d}/\mathrm{d}z$. It is now very tempting to try to consider (2.2) as the formal definition of a hyperdifferential operator. Since such a hyperdifferential operator is uniquely determined by its Fourier–Borel symbol, we shall try to compute the symbol of (2.2):

$$e^{-z\xi}\tilde{Q}F(A^*, A)\, e^{z\xi} = e^{-z\xi} \int \exp(w'z - z'(\mathrm{d}/\mathrm{d}z))e^{z\xi}\, \mathrm{d}\nu_F(z', w')$$

$$= e^{-z\xi} \int e^{w'z} \exp(-z'\,(\mathrm{d}/\mathrm{d}z)) \exp(\tfrac{1}{2}\, w'z')\, e^{z\xi}\, \mathrm{d}\nu_F(z', w')$$

$$= \int e^{w'z-z'\xi} \exp(\tfrac{1}{2}\, w'z')\, \mathrm{d}\nu_F(z', w').$$

Hence (still formally)

$$\sigma\tilde{Q}F(A^*, A)\,(z, \xi) = \exp(\tfrac{1}{2}\, \partial_z \partial_\xi) \int e^{w'z-z'\xi}\, \mathrm{d}\nu_F(z', w') = \exp(\tfrac{1}{2}\, \partial_z \partial_\xi)\, F(z, \xi). \tag{2.3}$$

Remark. In fact, we should use the correspondence $z \leftrightarrow \sqrt{\hbar}\, A^*$, $\xi \leftrightarrow \sqrt{\hbar}\, A$ instead of $z \leftrightarrow A^*$, $\xi \leftrightarrow A$. Indeed, z and ξ correspond to $(x - ip)/\sqrt{2}$, $(x + ip)\sqrt{2}$, respectively, (see [9, 10a]), which implies that the corresponding operators should have commutator h, whereas $[A, A^*] = 1$. The formal expression (2.3) should therefore be $\sigma\tilde{Q}F(z, \xi) = \exp(\frac{1}{2}\hbar\partial_z \partial_\xi)\, F(z, \xi)$.

In the following, we shall put $\hbar = 1$, with an exception for the twisted product (Section 3.1), where a power expansion in $\hbar$ will be made. Expression (2.3) is still formal, because we do not yet know whether the function defined in (2.3) is a nicely behaved function satisfying Condition (2) in Theorem 1.2. In the following theorem, we see that it does indeed satisfy this condition.

THEOREM 2.2. *For any a in* $\mathbb{R}$, *the application* $\Phi_a : F \to \exp(a\, \partial_z \partial_\xi)F$ *defines a bijective map from* Exp_2 *to itself, with inverse* Φ_{-a}.

(The proof is easy: using the Cauchy inequalities, and the boundedness of the series $n!n^{-n-1/2}e^n$, one obtains the desired result.)

In particular, for any F in Exp_2 the function $\sigma\tilde{Q}F$ defined by (21) is an element of Exp_2. Hence, it certainly satisfies Condition (2) in Theorem 1.2, which implies that $\sigma\tilde{Q}F$ determines uniquely a hyperdifferential operator $\tilde{Q}F(A^*, A)$. From now on, we shall take this $\tilde{Q}F$ (which is now rigorously defined) as our starting point. We are, however, ultimately interested in a correspondence between classical functions and operators on the Hilbert space $\mathcal{H}$, not on all of E. We have, thus, to investigate whether the hyperdifferential operator defined by this Fourier–Borel symbol can be restricted to a (possibly unbounded) operator on $\mathcal{H}$. This will be done in the next section.

2.1. RESTRICTION TO $\mathcal{H}$

We do not expect to be able to define a restriction of $\tilde{Q}F$ to $\mathcal{H}$ as a bounded operator. One sees immediately that, for instance, the derivation operator d/dz cannot be defined as a bounded operator on $\mathcal{H}$, although it is clearly a hyperdifferential operator on E. So we will have to be careful about domain problems. It seems reasonable to consider the set of all the elements f of $\mathcal{H}$ whose image $\tilde{Q}Ff$ is again an element of $\mathcal{H}$, and this set will, in fact, be the domain of our $\tilde{Q}F$ (see Definition 2.4). But first we will see that this set is dense, by proving that all the monomials, hence all the polynomials, are contained in it (the polynomials are dense in $\mathcal{H}$: the monomials $u_k(z) = (k!)^{-1/2}z^k$ form an orthonormal base of $\mathcal{H}$). This is certainly not true for a general hyperdifferential operator. If we take, for instance, the hyperdifferential operator A defined by $a_{mn} = (m!)^{-1/2}\delta_{n0}$, (these a_{mn} satisfy Condition (3) in Theorem 1.2, and define thus a hyperdifferential operator), then $(Au_0)(z) = \Sigma_m\ (m!)^{-1/2}z^m$ is not an element of $\mathcal{H}$. However, for our operators QF it does work:

THEOREM 2.3. *Let F be an element of* Exp_2, *and $u_k \in \mathcal{H}$ the monomial defined by $u_k(z) = (k!)^{-1/2}z^k$. Then, for any k in* $\mathbb{N}$: $\tilde{Q}F\, u_k \in \mathcal{H}$.

Sketch of Proof. One uses $|\partial\tilde{Q}F(z, \xi)| \leqslant M \exp(K(|z| + |\xi|))$ to derive the bound $|a_{mn}| \leqslant C(m!n!)^{-1}\ (K+1)^{m+1}$. It is then simply a matter of writing things out explicitly to see that Expression (1.2) converges for $\tilde{Q}Fu_k$.

DEFINITION 2.4. *We define the operator $\tilde{Q}F$ and its domain $D_2(F)$ by $D_2(F) = \{f \in \mathcal{H};\ \tilde{Q}Ff \in \mathcal{H}\}$; $QF = \tilde{Q}F|_{D_2(F)}$.*

This operator QF is a densely defined, but, in general, unbounded operator, although $\tilde{Q}F$ was a continuous operator on E. This is due to the fact that we are considering two different topologies: we have not only the usual Hilbert topology τ_2; as a subset of E, $\mathcal{H}$ can also be equipped with the relative topology $\tau_{E,\mathcal{H}}$. One can easily show that any sequence converging with respect to τ_2 converges also with respect to $\tau_{E,\mathcal{H}}$; since both topologies are metrizable, this implies that τ_2 is stronger than $\tau_{E,\mathcal{H}}$. Using this fact, we prove in the next theorem that the operators $\tilde{Q}F$ are closed.

THEOREM 2.5. *Let F be an element of* Exp_2. *The operator QF given by Definition 2.4 is closed.*

Proof. Let $(f_n)_n$ be a sequence in $D_2(F)$ such that both sequences $(f_n)_n$ and $(QFf_n)_n$ converge in τ_2; let f and g be their respective limits. Because of the preceding remark and the definition of QF, this implies $f_n \to f$ in τ_E; $\tilde{Q}Ff_n \to g$ in τ_E. Since $\tilde{Q}F$ is continuous with respect to τ_E, and since τ_E is a Haussdorff-topology, we have $g = \tilde{Q}Ff$. Hence, f is an element of $D_2(F)$, and $QFf = g$. □

3. SOME APPLICATIONS

3.1. THE HOLOMORPHIC TWISTED PRODUCT

Just as in the case of Weyl-quantization, we can define a twisted product in the holomorphic quantization procedure. The problem is to find for any two functions F, G in Exp_2 a function H such that $\tilde{Q}H = \tilde{Q}F \circ \tilde{Q}G$. We shall prove in the following theorem that this H indeed exists in Exp_2, and is unique. In fact, one transports the multiplication of the operator space to the classical function space. This multiplication turns the function space into a (non-commutative) algebra, and the mapping from functions to operators into an algebra-morphism (see, for instance, [8, 13, 14]). We shall not enter into all this and simply prove, as we said before, that a twisted product can be defined on the class of classical functions considered here. First we prove the following lemma:

LEMMA 3.1. *Let A, B be two hyperdifferential operators with Fourier–Borel symbols in* Exp_2. *Then the Fourier–Borel symbol of the product of these operators is again in* Exp_2.
Proof. There exist M, M', K, K' such that $|\hat{\sigma}A(z,\xi)| \leqslant M \exp(K(|z| + |\xi|))$ and $|\hat{\sigma}B(z,\xi)| \leqslant M' \exp(K'(|z| + |\xi|))$. Theorem 1.3 tells us that $\hat{\sigma}AB(z,\xi) = \Sigma_{n=0}^{\infty} (n!)^{-1}\ \partial_\xi^n \hat{\sigma}A(z,\xi) \times \times\ \partial_z^n \hat{\sigma}B(z,\xi)$. Applying the Cauchy inequalities yields, for any $R, S > 0$,

$$|\partial_\xi^n \hat{\sigma}A(z,\xi)\ \partial_z^n \hat{\sigma}B(z,\xi)| \leqslant MM'(n!)^2\ R^{-n} S^{-n} \exp(K(|z| + |\xi| + R)) \exp(K'(|z| + |\xi| + S)).$$

Putting $R = n(K+1)^{-1}$, $S = n(K'+1)^{-1}$, and using again the boundedness of the series $n!\, n^{-n-1/2} e^n$, we obtain

$$|\hat{\sigma}(AB)(z,\xi)| \leqslant MM'(K+1)(K'+1) M_1 \exp((K+1)(K'+1)) \exp((K+K')(|z| + |\xi|)) \quad \square$$

If F, G are elements of Exp_2, then $\hat{\sigma}\tilde{Q}F$, $\hat{\sigma}\tilde{Q}g$ are elements of Exp_2. From Lemma 3.1, we see that $\hat{\sigma}(\tilde{Q}F \circ \tilde{Q}G)$ is then also an element of Exp_2, which implies that $H = \exp(-\frac{1}{2}\ \partial_z \partial_\xi)\hat{\sigma}(\tilde{Q}F \circ \tilde{Q}G)$ is contained in Exp_2. Hence we have proven the first part of the following theorem:

THEOREM 3.2. *Let F, G be two elements of* Exp_2. *There exists a unique H in* Exp_2 *such that* $\tilde{Q}H = \tilde{Q}F \circ \tilde{Q}G$. *This H is given by:*

$$H(z,\xi) = \exp(-\tfrac{1}{2}\ \partial_z \partial_\xi)\ \{\exp(\partial_z \partial_\xi)\ [\exp(\tfrac{1}{2}\ \partial_{z_1} \partial_\xi) F(z_1, \xi) \exp(\tfrac{1}{2}\ \partial_z \partial_{\xi_1}) G(z, \xi_1)]\ \Big|_{\substack{z_1 = z \\ \xi_1 = \xi}} \}. \tag{3.1}$$

Proof. The existence of H is already proven. Unicity follows from Theorem 2.2, more specifically from the fact that the maps $\exp(\frac{1}{2}\ \partial_z \partial_\xi)$ are bijections from Exp_2 to itself. □

DEFINITION 3.3. *The univocally defined function H in Theorem 3.2 will be called the holomorphic twisted product of F and G; it will be denoted by* $H = F \circledast G$ *(see* [10]*).*

Theorem 3.2 states that $\tilde{Q}(F \circledast G) = \tilde{Q}F \circ \tilde{Q}G$ on E. We can restate this theorem in terms of unbounded operators on $\mathcal{H}$:

COROLLARY 3.4. *Let F, G be elements of* Exp_2. *The operator* $QF \cdot QG$ *with domain* $D_2(G) \cap (QG)^{-1}\ (D_2(F))$ *is densely defined, and there exists a unique H in* Exp_2 *such that* $QH \supset QF \cdot QG$. *This H is given by* $F \circledast G$.
Proof. Let H be the function $F \circledast G$. For any polynomial ψ we have $\psi \in D_2(G)$. On the other hand, we know from Theorem 2.3 that $\tilde{Q}H\psi \in \mathcal{H}$, hence $\tilde{Q}F(QG\psi) \in \mathcal{H}$, which implies that $QF \cdot QG$ is densely defined. The inclusion $QH \supset QF \cdot QG$ follows immediately. Since, moreover, the polynomials are dense in E with respect to τ_E, any hyperdifferential operator is uniquely determined by its restriction to the polynomials. Together with the unicity in Theorem 3.2, this yields the uniqueness of H.

Although well-defined, expression (3.1) is not very useful for practical purposes. It can be rewritten as follows (we introduce $\hbar$):

$$(F \circledast G)(z, \xi) = \exp(-\tfrac{1}{2}\hbar\partial_z\partial_\xi)\ \{\exp(\hbar\partial_z\partial_\xi)\ [\exp(\tfrac{1}{2}\hbar\partial_{z_1}\partial_\xi)F(z_1, \xi)\exp(\tfrac{1}{2}\partial_z\partial_{\xi_1})G(z, \xi_1)]\Big|_{\substack{z_1 = z\\ \xi_1 = \xi}}\}$$

$$= \iint d\nu_F(z', w')\, d\nu_G(z'', w'')\exp(z(w' + w'') - w(z' + z''))\exp(\tfrac{1}{2}\hbar(w'z'' - w''z')), \tag{3.2}$$

where ν_F, ν_G are the 'measures' defined by (2.1′).

Using (3.2), we can make a power expansion in $\hbar$:

$$(F \circledast G)(z, \xi) = \sum_{n=0}^{\infty} (2^n n!)^{-1}\hbar^n \mathscr{P}^n(F, G), \tag{3.3}$$

where $\mathscr{P}^n$ is the nth power of the 'holomorphic Poisson bracket':

$$\mathscr{P}(F, G) = \partial_z F \cdot \partial_w G - \partial_w F \cdot \partial_z G.$$

Expression (3.3) can now be used in applications. For the harmonic oscillator for instance, $H(z, \xi) = z\xi$, one can check that $H \circledast f(H) = H \cdot f(H) - (\hbar^2/4)(f' + Hf'')$. This is exactly relation (6.1) in [8b]; using an appropriate distribution space [15], we can now calculate $\exp((i/\hbar)Ht)$, and hence compute the spectrum $E_n = \hbar(n + 1/2)$ of the harmonic oscillator as it was done in [8b].

3.2. ADJOINTS

We shall see here another application of our approach. As a result of the fact that the operator QF, although unbounded, can be considered as restrictions of well-defined and continuous operators $\tilde{Q}F$, we shall be able to obtain some results about the adjoints of our operators in a very simple way. We shall proceed as we did in the beginning: first, we bluntly state what the formal adjoint of a QF should look like, and we compute its Fourier–Borel symbol. Then we take this symbol as our starting point for the following.

Let F be an element of Exp_2', with $\hat{\sigma}\tilde{Q}F(z, \xi) = \Sigma_{m,n}\, a_{mn} z^m \xi^n$. The operator QF is then formally given by $QF = \Sigma_{m,n}\, a_{mn} A^{*m} A^n$. Since A^* is the adjoint of A (see [11a]) the adjoint of QF is formally given by $\Sigma_{m,n}\, \overline{a_{mn}} A^{*n} A^m$. This operator corresponds to the Fourier–Borel symbol

$$\sum_{m,n} \overline{a_{mn}} z^n \xi^m = \overline{\hat{\sigma}\tilde{Q}F(\bar{\xi}, \bar{z})}.$$

So this Fourier–Borel symbol is our starting point. It is obvious that for F in Exp_2, the function F^* defined by $F^*(z, \xi) = \overline{F(\bar{\xi}, \bar{z})}$ is again an element of Exp_2. The corresponding Fourier–Borel symbol satisfies the relation $\hat{\sigma}\tilde{Q}F^*(z, \xi) = \overline{\hat{\sigma}\tilde{Q}F(\bar{\xi}, \bar{z})}$. We shall now investigate the relation between $(QF)^*$ and QF^*. To do this, we first prove the following lemma:

LEMMA 3.5. *Let F be an element of* Exp_2, *with* $\delta\tilde{Q}F(z,\xi) = \Sigma_{m,n=0}^{\infty} a_{mn} z^m \xi^n$; *let f be an element of $D_2(F)$ with power series expansion $f(z) = \Sigma_{l=0}^{\infty} f_l u_l(z)$. Then QFf is given by*

$$(\tilde{Q}Ff)(z) = \sum_{k=0}^{\infty} u_k(z) \left(\sum_{l=0}^{\infty} \sum_{m=0}^{\min(k,l)} f_l (l!k!)^{1/2} (m!)^{-1} a_{k-m,\, l-m} \right). \tag{3.4}$$

Proof. We introduce the notation $\psi_l(z) = z^l = (l!)^{1/2} u_l(z)$. Since $\tilde{Q}F$ is continuous on E, we have (the sums are defined with respect to τ_E)

$$\begin{aligned} \tilde{Q}Ff &= \sum_{l=0}^{\infty} (l!)^{-1/2} f_l \tilde{Q}F(\psi_l) \\ &= \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{l} f_l a_{mn} (l!)^{1/2} ((l-n)!)^{-1} \psi_{l+m-n}. \end{aligned}$$

Using the fact that $|a_{mn}| \leqslant MK^{m+n}(m!n!)^{-1}$ for some M, K, one can check that this series converges absolutely and uniformly on compact sets, which implies that we can arrange the summations to obtain expression (3.4). This series converges to $\tilde{Q}Ff$ with respect to τ_E; since it gives also the series expansion of an element of $\mathcal{H}$ (f is an element of $D_2(F)$), it converges to QFf in τ_2. □

With the help of this lemma we can prove the following:

PROPOSITION 3.6. *Let F be an element of* Exp_2. *Then $(QF)^* \subset QF^*$.*

Proof. Let g be an element of $D((QF)^*)$, with $(QF)^*g = h$; h has the series expansion $h = \Sigma_k h_k u_k$ with

$$h_k = (u_k, h) = (u_k, (QF)^*g) = (QFu_k, g) = \sum_{l=0}^{\infty} g_l \sum_{m=0}^{\min(k,l)} (l!k!)^{1/2} (m!)^{-1} \; a_{l-m,\, k-m}.$$

But this implies $h = QF^*g$. Since $h \in \mathcal{H}$, we have now $g \in D_2(F^*)$ and $QF^*g = (QF)^*g$. Hence $(QF)^* \subset QF^*$. □

Notice that this proof works for any restriction of QF to a domain containing the polynomials. This remark leads to the following corollary:

COROLLARY 3.7. *Let F be an element of* Exp_2; *let A be an operator on $\mathcal{H}$ such that $D(A)$ contains the polynomials and $A \subset QF$. Then $A^* \subset QF^*$.*

The restriction of QF to the polynomials will play an important role in the following. This is the reason why we introduce the next definition:

DEFINITION 3.8. *Let F be an element of* Exp_2. *The restriction of QF to the polynomials will be denoted by Q_pF.*

Corollary 3.7 guarantees us that $(Q_pF)^* \subset QF^*$. (3.5)

It is easy to see that the series obtained by substituting (3.4) in (f, QFg) is absolutely summable whenever f or g is a polynomial, which implies that in this case $(f, QFg) = (QF^*f, g)$. Hence, $QF^* \subset (Q_pF)^*$. Taking into account (3.5), this yields:

$$Q_pF \subset \overline{Q_pF} = (QF^*)^* \subset QF = (Q_pF^*)^*. \tag{3.6}$$

If F is symmetric, i.e. $F^* = F$, we see from (3.6) that $QF = (Q_pF)^*$; moreover, QF is selfadjoint iff Q_pF is essentially selfadjoint. Applying a well-known result (see, for instance, [16], Theorem VIII.3), we see that the essentially-selfadjointness of Q_pF is equivalent with $\mathrm{Ker}(QF \pm i) = \{0\}$. We can, however, consider $\tilde{Q}F$, which is a nice operator on E, instead of QF, and try to solve the equations $(\tilde{Q}F \pm i)f = 0$ in E. It is easy to see that $\mathrm{Ker}(QF \pm i) = \mathrm{Ker}(\tilde{Q}F \pm i) \cap \mathcal{H}$. Hence, QF is selfadjoint iff no non-zero solution of the equations $(\tilde{Q}F \pm i)f = 0$ is an element of $\mathcal{H}$. Using this criterion, one can prove, for instance, that for $F(z, \xi) = \exp(\alpha z + \bar{\alpha}\xi)$, where $\alpha \in \mathbb{C}$, the operator QF is selfadjoint, Indeed, if we put $u = (|\alpha|)^{-1}\alpha z$, then $\alpha z = |\alpha|u$ and $\alpha(\mathrm{d}/\mathrm{d}z) = |\alpha|(\mathrm{d}/\mathrm{d}u)$. This implies that QF is selfadjoint iff QG is selfadjoint, with $G(z, \xi) = \exp(|\alpha|(z + \xi))$. The calculation of $\bar{\partial}\tilde{Q}G$ yields $\bar{\partial}\tilde{Q}G(z, \xi) = \exp(\frac{1}{2}\ |\alpha|^2)$, $\exp(|\alpha|(z + \xi))$, hence, up to some constant factor, $\tilde{Q}G = \Sigma_{m,n=0}^{\infty}\ |\alpha|^{m+n}(m!n)^{-1} z^m (\mathrm{d}/\mathrm{d}z)^n$. One can now check that every solution of $\tilde{Q}Gf = if$ has the form

$$f(z) = \exp(-z^2/2)\exp(\tfrac{1}{2}\ (|\alpha| + i\pi/|\alpha|)\, z) \sum_{k \in \mathbb{Z}} \beta_k \exp(2i(\pi/|\alpha|)\, kz),$$

which implies that $|f(z)|^2 \exp(-|z|^2)$ is not integrable on $\mathbb{C}$ (considered as $\mathbb{R}^2$), hence that f is not an element of $\mathcal{H}$. Analogously, one proves $\{f \in \mathcal{H}; \tilde{Q}Ff = -if\} = \{0\}$; this implies then that Q_pF is essentially selfadjoint, i.e. that QF is selfadjoint. It is equally simple to check that, for instance, any polynomial with real coefficients in $(\alpha z + \bar{\alpha}\xi)$ yields a selfadjoint QF.

In general, however, the inclusion $Q_pF \subset (Q_pF)^* = QF$ cannot be replaced by an equality for symmetric $F^\dagger$. Here, one sees the limitations of the hyperdifferential operator approach: for these cases, additional domain specifications are necessary, which should come from a careful analysis of the form of the individual operators. However, every selfadjoint extension of Q_pF is a restriction of QF, which implies that we know explicitly from Lemma 3.5 how this extension works on its domain. (This is analogous to the case where formal differential operators on $L^2(\mathbb{R})$ are discussed: see [17].) On the other hand, Corollary 3.4 is still true if one considers restrictions of QF, QG: it is easy to check that for any F, G in Exp_2, there is a unique $H = F Ⓜ G$ in Exp_2 such that $Q_pH = QF \circ Q_pG$.

Remark. In this connection, it may be useful to remark that although Babbitt in [10] considers a bigger domain for his operators than only the polynomials, this domain yields exactly the same difficulties as the polynomials, in the sense that the restriction of QF to this domain is essentially selfadjoint iff Q_pF is essentially selfadjoint.

† One can check, for instance, that for $F_1(z, \xi) = (z + \xi)^4 - z\xi$ the operator Q_pF_1 is not essentially selfadjoint. (One way of checking this is to use the unitary correspondence of $\mathcal{H}$ and $L^2(\mathbb{R})$ given in [11a]; one sees then that Q_pF_1 is unitarily equivalent to the operator $(1/\overline{2})\,((\mathrm{d}^2/\mathrm{d}x^2) - x^2 + 2x^4 + 3)$ on the domain $\{f \in L^2(\mathbb{R}); f$ is a linear combination of Hermite functions$\}$.) On the other hand for $F_2(z, \xi) = (z + \xi)^4 + z\xi$, the operator Q_pF_2 is essentially selfadjoint; this shows that a deeper analysis than only the hyperdifferential operator approach is necessary to make a distinction between F_1 and F_2.

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