A CHARACTERIZATION OF SUBSYSTEMS IN PHYSICS

DIRK AERTS and INGRID DAUBECHIES*
Theoretische Natuurkunde, Vrije Universiteit Brussel, Pleinlaan 2, B-1050 Brussel, Belgium

ABSTRACT. Working within the framework of the propositional system formalism, we use a previous study [1] of the description of two independent physical systems as one big physical system to derive a characterization of a (non-interacting) physical subsystem. We discuss the classical case and the quantum case.

1. INTRODUCTION

We shall follow Piron [2] and describe any physical system by means of the collection of its properties, or, equivalently, of the yes–no experiments which can be carried out on this system. In [2], it is shown that this collection is a propositional system, that is a complete, orthocomplemented, weakly modular, atomic lattice satisfying the covering law. The states of the physical system are represented by the atoms of the lattice. For the definitions of these concepts and the physical justification of this approach, see [2] or also [1]. In what follows we shall use the abbreviation PROP for these propositional systems.

In [1], we studied the description of two non-interacting physical systems as one joint physical system. We denote these two independent systems by S_1, S_2, and the big physical system containing them both by S. The corresponding PROP's are $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}$. From a few simple arguments resulting from physical considerations, we arrived at the following structure (see [1], §2):

(1.1) There exist c-morphisms† h_1, h_2 from $\mathcal{L}_1$, $\mathcal{L}_2$ to $\mathcal{L}$ with $h_1(I_1) = I$, $h_2(I_2) = I$. (I_1, I_2 are the maximal elements in $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}$, respectively.) This is the mathematical translation of the fact that the structures of S_1 and of S_2 are conserved.

(1.2) For a_1 in $\mathcal{L}_1$, a_2 in $\mathcal{L}_2$, we have $h_1(a_1) \leftrightarrow h_2(a_2)$, ($h_1(a_1)$ and $h_2(a_2)$ are compatible). This is the mathematical formulation of the fact that S_1, S_2 are supposed to be independent.

(1.3) For p_1 atom in $\mathcal{L}_1$, p_2 atom in $\mathcal{L}_2$, we have that $h_1(p_1) \wedge h_2(p_2)$ is an atom in $\mathcal{L}$. This means that maximal information on S_1, S_2 separately yields maximal information on S: S_1, S_2 are the only constituents of S.

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*Wetenschappelijke medewerkers bij het Interuniversitair Instituut voor Kernwetenschappen (in het kader van navorsingsprogramma 21 EN).

†A c-morphism is a map conserving the complete, orthocomplemented, weakly modular lattice structure. When a c-morphism maps the maximal element onto the maximal element it is said to be unitary (see [2]).

Letters in Mathematical Physics 3 (1979) 11–17.
0377–9017/79/0031–0011 $00.70.
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With these three requirements we were able in [1] to prove some results about the PROP of the joint physical system, and this in both quantum and classical cases. Indeed, both classical and quantal systems can be described in the framework of the propositional approach. For the classical systems, one introduces one more property, namely distributivity, representing the well-known physical fact that in this case all possible experiments can be carried out independently of each other [2]. Such a distributive PROP can be shown to be isomorphic to $\mathscr{P}(\Omega)$, i.e. to the lattice (with respect to set-theoretic inclusion) of all the subsets of the set Ω of its atoms [2]. This set Ω is then called the phase space of the classical system. Using the three conditions mentioned above (in fact only the first and the third ones: the second one becomes redundant in this case) we proved that when a classical physical system S is constituted by two classical systems S_1, S_2 with respective phase spaces Ω_1, Ω_2, its phase space is given by $\Omega_1 \times \Omega_2$ [1].

When the PROP is not distributive, we make a distinction between pure quantum systems and more general systems. A pure quantum system has no classical features, i.e. no superselection rules: there does not exist any yes–no experiment compatible with all the others. When this is the case, we say the PROP is irreducible. It is proven in [3] that any such irreducible PROP (granted that it contains at least four orthogonal atoms) is isomorphic to the lattice of all biorthogonal subspaces of some vectorspace V, $\mathbb{K}$, where the orthogonality is defined with respect to some sesquilinear form on V, $\mathbb{K}$, and where $F^{\perp} + F^{\perp\perp} = V$ for any subspace F of V. This structure looks quite formidable, but it is not really so terrifying. If one takes the field $\mathbb{K}$ to be $\mathbb{C}$, one can prove [3], [4] that the structure is exactly the one encountered in the usual quantum formalism: the lattice described above becomes now the lattice of all closed subspaces of a complex Hilbert space $\mathcal{H}$, or, equivalently, the lattice of all projection operators in this Hilbert space. The atoms (= states) are then given by the one-dimensional subspaces of $\mathcal{H}$.

When there exist superselection rules, the PROP can be considered as a combination of pure quantal propositional systems [2]. We will not consider such composite systems here.

Applying our three conditions stated above to this setting, we proved in [1] that when a physical quantum system S is made up of two pure quantum systems S_1, S_2 with respective Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$, it is described by the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ or $\mathcal{H}_1^{\dagger} \otimes \mathcal{H}_2$. Since our three conditions proved to be sufficient to derive the usual coupling procedures for the simultaneous description of two independent physical systems, it is a natural question to ask whether they can also be used to characterize physical subsystems of a big physical system. How this is done will be explained in the next section.

2. CHARACTERIZATION OF A PHYSICAL SUBSYSTEM

Our aim is here to investigate the conditions under which a sublattice $\tilde{\mathcal{L}}$ of the PROP $\mathcal{L}$ of a physical system S can be considered as the PROP of a physical subsystem $\tilde{S}$ of S. In other words, given $\tilde{\mathcal{L}} \subset \mathcal{L}$, we want to be able to ascertain whether $\mathcal{L}_1$, $\mathcal{L}_2$, h_1, h_2 exist, satisfying conditions (1.1), (1.2), (1.3) and for which

$$h_1(\mathcal{L}_1) = \tilde{\mathcal{L}}. \tag{1.4}$$

This motivates the following definition:

2.1. DEFINITION. *A sublattice $\tilde{\mathcal{L}}$ of the PROP $\mathcal{L}$ of a physical system S is said to represent a physical subsystem of S iff there exist PROP's $\mathcal{L}_1$, $\mathcal{L}_2$ and maps h_1, h_2 satisfying* (1.1), (1.2), (1.3) *and* (1.4).

To carry out our investigation, we make a preliminary study of the situation discussed in [1]. Let the physical systems S_1, S_2 be constituents of a big physical system S, then we have unitary c-morphisms h_1, h_2 mapping $\mathcal{L}_1$, $\mathcal{L}_2$ to $\mathcal{L}$ ($\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}$ are the PROP's of S_1, S_2, S). One can easily check that $\mathcal{L}_1$, $\mathcal{L}_2$ are isomorphic to their images $h_1(\mathcal{L}_1)$, $h_2(\mathcal{L}_2)$: condition (1.3) insures that *f* or any p, q atoms in $\mathcal{L}_1$, $\mathcal{L}_2$, $h_1(p)$ and $h_2(q)$ are non-void, from which we infer that $\ker h_1 = \{O_1\}$, $\ker h_2 = \{O_2\}$ or, equivalently, that both h_1 and h_2 are injective (O_1, O_2 are the minimal elements in $\mathcal{L}_1$, $\mathcal{L}_2$). We focus our attention on $h_1(\mathcal{L}_1)$. Since it is isomorphic to $\mathcal{L}_1$, $h_1(\mathcal{L}_1)$ is a PROP, embedded in $\mathcal{L}$. The operations $\wedge$, $\vee$ (see [2]) on the PROP $h_1(\mathcal{L}_1)$ are the restrictions to $h_1(\mathcal{L}_1)$ of $\wedge$, $\vee$ defined on $\mathcal{L}$; the orthocomplementation on $h_1(\mathcal{L}_1)$ is the relative orthocomplementation induced by $\mathcal{L}$:

$$h_1(a)^r = h_1(a') = h_1(I_1) \wedge h_1(a)'.$$

Whenever a PROP $\tilde{\mathcal{L}} \subset \mathcal{L}$ has this structure, we call it a sub-PROP of $\mathcal{L}$ (see also [6], where it is called a propositional subsystem). Since h_1 is unitary, we have $h_1(I_1) = I$. So if $\tilde{\mathcal{L}} \subset \mathcal{L}$ is the PROP representing a physical subsystem $\tilde{S}$ of S, $\tilde{\mathcal{L}}$ has to be a PROP of $\mathcal{L}$ containing the identity I.

These conditions are however not necessarily sufficient: it may happen that although $\tilde{\mathcal{L}}$ has this nice structure, no two maps h_1, h_2 can be found satisfying (1.1) – (1.4) for some $\mathcal{L}_1$, $\mathcal{L}_2$. In this case, $\tilde{\mathcal{L}}$ is a sub-PROP of $\mathcal{L}$, but this fact has no direct physical consequences in this context: $\tilde{\mathcal{L}}$ is not the PROP, embedded in $\mathcal{L}$, of a physical subsystem $\tilde{S}$ of S. From this point onwards, we will examine the classical case and the pure quantum case separately.

2A. *The Classical Case*

In this case, $\mathcal{L}$ can be taken to be $\mathscr{P}(\Omega)$. From the construction given in [1], we easily see that if $\tilde{\mathcal{L}}$ is the representation of a physical subsystem of S, the sub-PROP's $\mathscr{P}(p)$, where the p's are the atoms of $\tilde{\mathcal{L}}$, are isomorphic as sublattices of $\mathscr{P}(\Omega)$. Indeed,

$$h_1 : \mathscr{P}(\Omega_1) \to \mathscr{P}(\Omega_1 \times \Omega_2)$$

$$G_1 \mapsto \{(x_1, x_2); x_1 \in G_1, x_2 \in \Omega_2\},$$

hence

$$\phi_{u,v}: \mathscr{P}(h_1(\{u\})) \to \mathscr{P}(h_1(\{v\}))$$

$$\{u\} \times G_2 \mapsto \{v\} \times G_2,$$

is a c-morphism from the sub-PROP $\mathscr{P}(h_1(\{u\}))$ of $\mathscr{P}(\Omega)$ to $\mathscr{P}(h_1(\{v\}))$. So if $\tilde{\mathfrak{L}}$ represents a subsystem $\tilde{S}$ of S, the sub-PROP's $\mathscr{P}(p)$ of $\mathfrak{L}$, where the p's are the atoms of $\tilde{\mathfrak{L}}$, have to be isomorphic. (One can check that these sub-PROP's $\mathscr{P}(p)$ are isomorphic iff the atoms p, considered as subsets of $\mathscr{P}(\Omega)$, have the same cardinality.) The following theorem states that this condition is sufficient.

2.2. THEOREM. *Let $\tilde{\mathfrak{L}}$ be a PROP of a $\mathscr{P}(\Omega)$ containing the identity Ω. Then $\tilde{\mathfrak{L}}$ is the representation of a physical subsystem iff $\forall p$, q atoms in $\tilde{\mathfrak{L}}$: $\exists \phi_{p,q}$ c-isomorphisms from $\mathscr{P}(p)$ to $\mathscr{P}(q)$.*

Proof. The "only if" part is already proven. Take now one atom p in $\tilde{\mathfrak{L}}$. Define $\Omega_1 = \{q; q$ atom in $\tilde{\mathfrak{L}}\}$, $\Omega_2 = p$

$$h_1 : \mathscr{P}(\Omega_1) \to \mathscr{P}(\Omega)$$

$$G_1 \mapsto \bigcup_{q \in G_1} q$$

$$h_2 : \mathscr{P}(\Omega_2) \to \mathscr{P}(\Omega)$$

$$G_2 \mapsto \bigcup_{q \in A(\tilde{\mathfrak{L}})} \phi_{qp}(G_2)$$

(we have denoted by $A(\tilde{\mathfrak{L}})$ the set of atoms in $\tilde{\mathfrak{L}}$).

It is trivial to check that these maps are c-morphisms.

Moreover

$$h_1(\Omega_1) = \bigcup_{q \in \Omega_1} q = \bigcup_{q \in A(\tilde{\mathfrak{L}})} q = \Omega,$$

$$h_2(\Omega_2) = \bigcup_{q \in A(\tilde{\mathfrak{L}})} \phi_{qp}(p) = \bigcup_{q \in A(\tilde{\mathfrak{L}})} q = \Omega,$$

and

$$h_1(p_1) \cap h_2(p_2) = p_1 \cap \left(\bigcup_{q \in A(\tilde{\mathfrak{L}})} \phi_{qp}(p_2) \right)$$

$$= p_1 \cap \phi_{p_1 p}(p_2)$$

$$= \phi_{p_1 p}(p_2) \in A(\mathfrak{L}).$$

Hence $\mathscr{P}(\Omega_1)$, $\mathscr{P}(\Omega_2)$, h_1, h_2 satisfy conditions (1.1), (1.3). ((1.2) is trivially verified: $\mathscr{P}(\Omega)$ is distributive.) Moreover it is obvious that $h_1(\mathscr{P}(\Omega_2)) = \tilde{\mathfrak{L}}$, which completes the proof of the "if" part. □

2.B. *The Quantum Case*

If the physical system S is a pure quantum system, its PROP $\mathcal{L}$ can be taken to be a $\mathcal{P}(\mathcal{H})$. Suppose that the PROP $\tilde{\mathcal{L}}$ of $\mathcal{L}$ represents a physical subsystem $\tilde{S}$ of S. Since one can show that the coupling of a reducible PROP with another PROP yields again a reducible PROP†, $\tilde{S}$ is again a pure quantum system. Hence its PROP is isomorphic to a $\mathcal{P}(\tilde{\mathcal{H}})$. So a sub-PROP $\tilde{\mathcal{L}}$ of $\mathcal{L}$ represents a physical subsystem $\tilde{S}$ of S only if it is isomorphic to a $\mathcal{P}(\tilde{\mathcal{H}})$. This condition is however not sufficient, as is shown in the next theorem:

2.3. THEOREM. *Let $\mathcal{P}(\mathcal{H})$ be the PROP of the pure quantum system S, let $\tilde{\mathcal{L}}$ be a sub-PROP of $\mathcal{P}(\mathcal{H})$ such that:*

$I_{\mathcal{P}(\mathcal{H})} \in \tilde{\mathcal{L}}$,

$\exists\, \tilde{\mathcal{H}}$, *complex Hilbert space with* $\dim \tilde{\mathcal{H}} \geqslant 3$,

$\exists \phi\colon \mathcal{P}(\tilde{\mathcal{H}}) \to \tilde{\mathcal{L}}$, *c-isomorphism.*

Let i be the canonical injection from $\tilde{\mathcal{L}}$ to $\mathcal{P}(\mathcal{H})$. Then $\tilde{\mathcal{L}}$ is the representation of a physical subsystem of $\mathcal{P}(\mathcal{H})$ iff $i \circ \phi$ is a pure m-morphism.

Remark. We will use here some results about a special kind of c-morphisms, named m-morphisms (see [1, 5, 7]). If $\mathcal{H}'$, $\mathcal{H}''$ are two Hilbert spaces, a c-morphism f from $\mathcal{P}(\mathcal{H}')$ to $\mathcal{P}(\mathcal{H}'')$ is called an m-morphism iff $f(\overline{x+y}) \subset f(\bar{x}) + f(\bar{y})$ for any x, y in $\mathcal{H}'$ (the symbol $\bar{x}$ denotes the one-dimensional subspace $\mathbb{C}x$). Such an m-morphism can be shown to be generated by a family of linear or antilinear isometric maps ϕ_j from $\mathcal{H}'$ to $\mathcal{H}''$, such that the different $\phi_j(\mathcal{H}')$ are orthogonal and $\mathcal{H}'' = \bigoplus_{j \in J} \phi_j(\mathcal{H}')$. If all these maps are linear, resp. antilinear, f is called a linear, resp. antilinear, m-morphism. If both linear and antilinear maps occur in the decomposition, the m-morphism is mixed. A non-mixed m-morphism will be called a pure m-morphism.

Proof of the Theorem. We prove first the "only if" part. Since $\tilde{\mathcal{L}}$ is the representation of a physical subsystem, we have a unitary c-morphism h_1 mapping a $\mathcal{P}(\mathcal{H}_1)$ to $\mathcal{P}(\mathcal{H})$ such that $h_1(\mathcal{P}(\mathcal{H}_1)) = \tilde{\mathcal{L}}$. It was proven in [1] (§4, Lemma 4) that this h_1 is a pure m-morphism. Moreover $f = h_1^{-1} \circ i \circ \phi\colon \mathcal{P}(\tilde{\mathcal{H}}) \to \mathcal{P}(\mathcal{H}_1)$ is a subjective c-morphism mapping atoms to atoms, hence (see [5], Theorem 4.1) an m-morphism generated by a unitary or anti-unitary map Φ:

$$\Phi\colon \tilde{\mathcal{H}} \to \mathcal{H},$$

$$f(A) = \Phi(A) \qquad \forall A \in \mathcal{P}(\tilde{\mathcal{H}}).$$

Let $(\psi_j)_{j \in J}$ be a family isometric maps generating h_1. Then, $\forall G \in \mathcal{P}(\mathcal{H})$:

$$(i \circ \phi)(G) = (h_1 \circ f)(G) = h_1(\Phi(G)) = \bigoplus_{j \in J} (\psi_j \circ \Phi)(G).$$

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†If $\mathcal{L}_1$ is reducible, one can write it as a direct union or irreducible PROP's $(\mathcal{L}_{1,i})_{i \in I}$. The coupling procedure with $\mathcal{L}_2$ can then be applied for each of these $\mathcal{L}_{1,i}$ to obtain a $\mathcal{L}_i$. The joint PROP $\mathcal{L}$ is taken to be the direct union of these $\mathcal{L}_i$'s, and is thus again reducible (for the definition of reducibility and the here mentioned decomposition, see [2]).

The maps $\psi_j \circ \Phi$ are isometric; so $i \circ \Phi$ is generated by a family isometric maps $(\psi_j \circ \Phi)_{j \in J}$, which implies $i \circ \phi$ is an m-morphism (see [7]). Moreover, since h_1 *is a pure* m-morphism, these isometries are either all linear or all anti linear, which implies that $i \circ \Phi$ is a pure m-morphism (see [5]).

Let us now prove the "if" part. To do this, we will use some results obtained in the proof of Theorem 3.3 (more specifically Lemma 3.5) in [6]. There we proved, from the same suppositions concerning $\mathcal{H}, \tilde{\mathcal{L}}, \tilde{\mathcal{H}}, \phi$ as made here, that, for some arbitrary atom P in $\tilde{\mathcal{L}}$, a Hilbert space $\hat{\mathcal{H}}$, and an isomorphism $f: \mathcal{H} \to \hat{\mathcal{H}} \otimes P\mathcal{H}$ exist such that the isomorphism

$$\Phi: \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\hat{\mathcal{H}} \otimes P\mathcal{H})$$

$$A \mapsto f \circ A \circ f^{-1}$$

satisfies $\Phi(\tilde{\mathcal{L}}) = \mathcal{P}(\hat{\mathcal{H}}) \otimes \mathbb{1}_{P\mathcal{H}}$. (We consider here $\mathcal{P}(\mathcal{H})$, $\mathcal{P}(\hat{\mathcal{H}})$ as sets of projection operators in instead of closed subspaces.) If we take now $\mathcal{H}_1 = \hat{\mathcal{H}}$, $\mathcal{H}_2 = P\mathcal{H}$,

$$\phi_1 : \mathcal{P}(\mathcal{H}_1) \to \mathcal{P}(\mathcal{H}): Q \mapsto \Phi^{-1}(Q \otimes \mathbb{1}_{P\mathcal{H}}),$$

$$\phi_2 : \mathcal{P}(\mathcal{H}_2) \to \mathcal{P}(\mathcal{H}): R \mapsto \Phi^{-1}(\mathbb{1}_{\hat{\mathcal{H}}} \otimes R),$$

then it is obvious that ϕ_1, ϕ_2 satisfy the three conditions (1.1) – (1.3). Moreover

$$\phi_1(\mathcal{P}(\mathcal{H}_1)) = \Phi^{-1}(\mathcal{P}(\hat{\mathcal{H}}) \otimes \mathbb{1}_{P\mathcal{H}})$$

$$= \Phi^{-1}(\Phi(\tilde{\mathcal{L}})) = \tilde{\mathcal{L}},$$

hence condition (1.4) is fulfilled, which implies $\tilde{\mathcal{L}}$ represents a physical subsystem of S. □

Remarks. (1) As a consequence of this proof, we see that if $\tilde{\mathcal{L}}$ represents a physical subsystem, then the representation of the other constituent is given by:

$$h_2(\mathcal{P}(\mathcal{H}_2)) = \Phi^{-1}(\mathbb{1}_{\mathcal{H}} \otimes \mathcal{P}(P\mathcal{H})) = \Phi^{-1}(P((\mathcal{P}(\hat{\mathcal{H}}) \otimes \mathbb{1}_{P\mathcal{H}})'))$$

$$= P((\Phi^{-1}(\mathcal{P}(\hat{\mathcal{H}}) \otimes \mathbb{1}_{P\mathcal{H}}))') = P(\tilde{\mathcal{L}}').$$

(For any von Neumann algebra $\mathcal{R}$, we denote by $P(\mathcal{R})$ the set of the projection operators in $\mathcal{R}$).

(2) Since the conditions in Theorem 4.2 are exactly the same as in [6], Theorem 3.3, the following corollary holds:

COROLLARY. *Let $\mathcal{P}(\mathcal{H})$ be the PROP of the pure quantum system S, let $\tilde{\mathcal{L}}$ be a sub-PROP of $\mathcal{P}(\mathcal{H})$ containing the identity, such that $\tilde{\mathcal{L}}$ is isomorphic to a $\mathcal{P}(\tilde{\mathcal{H}})$ with* $\dim \tilde{\mathcal{H}} \geqslant 3$. *Then $\tilde{\mathcal{L}}$ represents a physical subsystem of $\tilde{S}$ of S iff $P(\tilde{\mathcal{L}}'') = \tilde{\mathcal{L}}$.*

ACKNOWLEDGMENTS

We would like to thank Professor C. Piron for his constant interest and Professor J. Reignier for his encouragement.

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(Received July 23, 1978)