A MATHEMATICAL CONDITION FOR A SUBLATTICE OF A PROPOSITIONAL SYSTEM TO REPRESENT A PHYSICAL SUBSYSTEM, WITH A PHYSICAL INTERPRETATION

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ABSTRACT. We display three equivalent conditions for a sublattice, isomorphic to a $\mathscr{P}(\tilde{\mathcal{H}})$, of the propositional system $\mathscr{P}(\mathcal{H})$ of a quantum system to be the representation of a physical subsystem (see [1]). These conditions are valid for dim $\tilde{\mathcal{H}} \geq 3$. We prove that one of them is still necessary and sufficient if dim $\tilde{\mathcal{H}} < 3$. A physical interpretation of this condition is given.

INTRODUCTION

Working in the propositional formalism, we studied in [2] the simultaneous description of two physical systems. On this base we gave in [1] a characterization of the propositional system (PROP) of a physical subsystem, when considered as a sublattice of the PROP of the big system. All this was done for both classical and quantum systems. For quantum systems in particular, we showed that a connection exists with results proven in [5].

Using again results proven in [5], we give here three different necessary and sufficient conditions for a sub-PROP of a PROP $\mathscr{P}(\mathcal{H})$ to represent a physical subsystem. The theorems of [5] we shall use here are only valid, however, when the sub-PROP contains at least three orthogonal atoms; hence our conditions are necessary and sufficient only under this restriction. Discussing the case where the restriction does not hold, we find that one of the three conditions is still necessary and sufficient. We have thus a generally valid criterion to decide whether a sub-PROP represents a physical subsystem. Moreover, a close scrutiny of this criterion leads to a physical interpretation which is completely consistent with our search for physical subsystems.

We review some definitions and previous results in Section 1. In Section 2, we give the three equivalent conditions; Section 3 contains the discussion of the case where the sub-PROP does not contain three orthogonal atoms. Finally, a physical interpretation of the one condition valid in the two cases is given in Section 4.

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1. DEFINITIONS AND PREVIOUS RESULTS

In the propositional formalism, a physical system is described by the propositional system (PROP) of its properties, i.e. by a complete, orthocomplemented, weakly modular, atomic lattice satisfying the covering law [3]. In particular, a pure quantum system is described by an irreducible PROP: no superselection rules exist [3]. Such an irreducible PROP can be shown to be isomorphic to the lattice of all closed subspaces of a complex Hilbert space [3]†. (Notation: $\mathcal{P}(\mathcal{H})$).

To study the joint description of two systems in one PROP, or to characterize sublattices as representing physical subsystems, we need a tool to compare different PROP's. This tool are the c-morphisms: maps from one orthocomplemented, complete lattice to another, which preserve the structure [3]. In the case where c-morphisms from one $\mathcal{P}(\mathcal{H})$ to a $\mathcal{P}(\mathcal{H}')$ are studied, one can consider a special class of c-morphisms, namely the m-morphisms (see [4], [6]). An m-morphism f from $\mathcal{P}(\mathcal{H})$ to $\mathcal{P}(\mathcal{H}')$ is a c-morphism for which $f(\bar{x}) + f(\bar{y})$ is a closed subspace of $\mathcal{H}'$ for any x, y in $\mathcal{H}$ (we use the symbol $\bar{x}$ to denote the closed subspace $\mathbb{C}x$). One can show that any such m-morphism f is generated by a family of isometric maps $(\phi_j)_{j\in J}$ with orthogonal images:

$$\forall j \in J : \phi_j : \mathcal{H} \to \mathcal{H}' \quad \text{isometric}$$

such that

$$\phi_i(\mathcal{H}) \perp \phi_j(\mathcal{H}), \qquad \text{if } i \neq j$$

$$\bigoplus_{j\in J} \phi_j(\mathcal{H}) = \mathcal{H}',$$

$$\forall G \in \mathcal{P}(\mathcal{H}) : f(G) = \bigoplus_{j\in J} \phi_j(G), \qquad \text{(see [4], [6]).}$$

If these ϕ_j are either all linear or all anti-linear, the m-morphism is called pure; otherwise, the m-morphism is mixed [1], [4].

Using an analysis made in [2], we gave in [1] the following characterization for a physical subsystem:

1.1. DEFINITION. *Let $\mathcal{L}$ be a PROP, $\tilde{\mathcal{L}}$ a sublattice of $\mathcal{L}$. Then $\tilde{\mathcal{L}}$ is said to represent a physical subsystem iff*

$\exists \mathcal{L}_1, \mathcal{L}_2$ *PROP*'s,

$\exists h_1, h_2$ *maps from $\mathcal{L}_1, \mathcal{L}_2$, respectively, to $\mathcal{L}$,*

such that:

(1.1) h_1, h_2 *are c-morphisms with* $h_1(\mathrm{I}_1) = \mathrm{I} = h_2(\mathrm{I}_2)$. ($\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}$ *are the maximal elements in* $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}$.)

†There is one difficulty about this. at some point in the discussion of irreducible PROP's, a field has to be introduced. If one takes this field to be $\mathbb{C}$, the result stated above is obtained.

(1.2) $\forall a_1 \in \mathcal{L}_1, a_2 \in \mathcal{L}_2 : h_1(a_1) \leftrightarrow h_2(a_2)$.
(The symbol $\leftrightarrow$ is used to denote compatibility; for the definition of compatibility, see [3] or also [2], [4])

(1.3) *If p_1 is an atom in $\mathcal{L}_1$, and p_2 an atom in $\mathcal{L}_2$, then $h_1(p_1) \wedge h_2(p_2)$ is an atom in $\mathcal{L}$.*

(1.4) $h_1(\mathcal{L}_1) = \tilde{\mathcal{L}}$.

In [1], we showed that conditions (1.1), (1.3) and (1.4) ensure that $\tilde{\mathcal{L}}$ is c-isomorphic to $\mathcal{L}_1$, hence not only a sublattice, but a sub-PROP of $\mathcal{L}$ (see [1], [5]). Condition (1.1) ensures that this sub-PROP contains the identity I. If we specialize to pure quantum cases, $\mathcal{L}$ can be taken to be a $\mathcal{P}(\mathcal{H})$, and $\tilde{\mathcal{L}}$ a sub-PROP of $\mathcal{P}(\mathcal{H})$, isomorphic to some $\mathcal{P}(\tilde{\mathcal{H}})$ [1]. We proved the following theorem in [1]:

1.2. THEOREM. *Let $\tilde{\mathcal{L}}$ be a sub-PROP of $\mathcal{P}(\mathcal{H})$, with $I_{\mathcal{P}(\mathcal{H})} \in \tilde{\mathcal{L}}$. Suppose that there exists an $\tilde{\mathcal{H}}$, complex Hilbert space with dimension $\geqslant 3$, and a φ, c-isomorphism from $\mathcal{P}(\tilde{\mathcal{H}})$ to $\mathcal{L}$. Let i be the canonical injection from $\tilde{\mathcal{L}}$ to $\mathcal{P}(\mathcal{H})$. Then $\mathcal{L}$ represents a physical subsystem iff the c-morphism $i \circ \varphi$ is a pure m-morphism.*

To prove this theorem we used some results of [5]. The criterion given by this theorem is however not quite so handy: for possible applications, other criteria might prove to be easier to handle. Such other criteria are given in the next section.

2. THREE EQUIVALENT CONDITIONS

In [5], we studied the connection between von Neumann algebras and sub-PROP's of a $\mathcal{P}(\mathcal{H})$. (We consider here $\mathcal{P}(\mathcal{H})$ as a set of projection operators in $\mathcal{P}(\mathcal{H})$, not as the set of closed subspaces of $\mathcal{H}$.) To be more precise, we investigated whether a sub-PROP $\mathcal{L}$ of $\mathcal{P}(\mathcal{H})$ contained already all the projection operators of the von Neumann algebra $\mathcal{L}''$ it generated. The following theorems were proven (see [5], Theorems 3.3 and 3.7):

2.1. THEOREM. *Let $\tilde{\mathcal{L}}$ be a sub-PROP of $\mathcal{P}(\mathcal{H})$ such that*

$\mathbb{1}_{\mathcal{H}} \in \tilde{\mathcal{L}}$.

$\exists \tilde{\mathcal{H}}$ *complex Hilbert space, with* $\dim \tilde{\mathcal{H}} \geqslant 3$, *and a c-isomorphism φ from $\mathcal{P}(\tilde{\mathcal{H}})$ onto $\tilde{\mathcal{L}}$. Let i be the canonical injection from $\tilde{\mathcal{L}}$ to $\mathcal{P}(\tilde{\mathcal{H}})$. Then $\tilde{\mathcal{L}} = P(\tilde{\mathcal{L}}'')$ iff $i \circ \varphi$ is a pure m-morphism.*

2.2. THEOREM. *Let $\mathcal{H}$, $\tilde{\mathcal{L}}$, $\tilde{\mathcal{H}}$, φ be as in Theorem 2.1. Then $\tilde{\mathcal{L}} = P(\tilde{\mathcal{L}}'')$ iff $\forall P, Q, R$ atoms in $\tilde{\mathcal{L}}$: $\exists \alpha \in \mathbb{C}$ such that $PQRP = \alpha P$.*

Taking this together with Theorem 1.2, we obtain the following result:

2.3. THEOREM. *Let $\mathcal{H}$, $\tilde{\mathcal{L}}$, $\tilde{\mathcal{H}}$, φ, i be as in Theorem 2.1. Then the following four statements are equivalent:*

(2.1) *$\tilde{\mathcal{L}}$ represents a physical subsystem.*

(2.2) $i \circ \varphi$ *is a pure m-morphism.*
(2.3) $P(\mathcal{L}'') = \mathcal{L}$.
(2.4) $\forall P, Q, R \in A(\tilde{\mathcal{L}}) : \exists \alpha \in \mathbb{C}$ *such that* $PQRP = \alpha P$.

(We have used the symbols $P(\mathcal{R})$ to denote the set of the projection operators in a von Neumann algebra $\mathcal{R}$, and $A(\mathcal{L})$ to denote the set of atoms of a PROP $\mathcal{L}$).

Hence we have now three equivalent criteria to decide whether $\mathcal{L}$ represents a physical subsystem: (2.2) was found in [1], and is the one which is closest to the approach in [2]. Condition (2.3) was already mentioned in [1], and gives us a connection between subsystems and von Neumann algebras. The third one, (2.4), seems to be the most practical one of the three.

One should not forget, however, that Theorem 2.3 only holds when $\dim \tilde{\mathcal{H}} \geqslant 3$. In the case where $\dim \tilde{\mathcal{H}} = 1$, everything becomes trivial. When, however, $\dim \tilde{\mathcal{H}} = 2$, we pointed out in [4] that the denomination 'pure m-morphism' is not relevant any more, since in this case not every m-morphism is a combination of a linear m-morphism and an antilinear one (see [4], §3, §5). So condition (2.2) should be dropped when $\dim \tilde{\mathcal{H}} = 2$. What becomes of the other two criteria is shown in the next section.

3. THE SPECIAL CASE DIM $\tilde{\mathcal{H}} = 2$

Having dropped condition (2.2), we are left with three different statements: (2.1), (2.3) and (2.4). Eventually, we will see that (2.3) is now stronger than the other two, which remain equivalent. This is the content of the following theorem.

3.1. THEOREM. *Let $\tilde{\mathcal{L}}$ be a sub-PROP of a $\mathcal{P}(\mathcal{H})$, containing $\mathbb{1}_{\mathcal{H}}$. Suppose that there exists a c-morphism φ from $\tilde{\mathcal{L}}$ to $\mathcal{P}(\tilde{\mathcal{H}})$, where $\tilde{\mathcal{H}}$ is a two-dimensional complex Hilbert space. Then the following implication diagramma holds:*

$$\begin{array}{ccc} (2.1) & \overset{\Leftarrow}{\not\Rightarrow} & (2.3) \\ & \Leftrightarrow\!\!\searrow \quad \swarrow\!\!\not\nearrow & \\ & (2.4) & \end{array}$$

Proof. (2.4) ⇒ (2.1). We already mentioned in [5] that whenever (2.4) holds, we can construct, for any P, Q in $A(\tilde{\mathcal{L}})$, partial isometries $\tilde{U}_{PQ}$ with initial subspace $Q\mathcal{H}$ and final subspace $P\mathcal{H}$, such that any of these $\tilde{U}_{PQ}$ can be written, up to some constant factor, as a product of elements in $A(\tilde{\mathcal{L}})$. These $\tilde{U}_{PQ}$, moreover, have the property that for any $P, Q, R \in A(\tilde{\mathcal{L}})$, there exists a $\beta \in C$ such that $\tilde{U}_{PQ}\tilde{U}_{QR} = \beta\tilde{U}_{PR}$.

Choose now one atom P in $\mathcal{L}$. Applying Proposition 5 in [7] I, §2, one can check (cf. [5]) that there exists a Hilbert space $\hat{\mathcal{H}}$ and an isomorphism ψ

$$\psi : \mathcal{H} \to \hat{\mathcal{H}} \otimes P\mathcal{H}$$

such that the isomorphism

$$\Psi: \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\hat{\mathcal{H}} \otimes P\mathcal{H})$$
$$A \mapsto \psi \circ A \circ \psi^{-1}$$

maps $\tilde{\mathcal{L}}$ into $\mathcal{P}(\hat{\mathcal{H}}) \otimes \mathbb{1}_{P\mathcal{H}}$. From the construction in [7], it is obvious that $\dim \hat{\mathcal{H}} = 2$; hence it is easy to check that for any atom Q in $\tilde{\mathcal{L}}$, there exists a p_Q atom in $\mathcal{P}(\hat{\mathcal{H}})$, such that $\Psi(Q) = p_Q \otimes \mathbb{1}_{P\mathcal{H}}$. Put now

$$\mathcal{H}_1 = \tilde{\mathcal{H}}, \quad \varphi_1 = \varphi, \quad \mathcal{H}_2 = P\mathcal{H}, \quad \varphi_2 : \mathcal{P}(\mathcal{H}_2) \to \mathcal{P}(\mathcal{H})$$
$$b \to \Psi^{-1}(\mathbb{1}_{\hat{\mathcal{H}}} \otimes b).$$

Since

$$\Psi \circ \varphi_1(\mathcal{P}(\mathcal{H}_1)) \subset \mathcal{P}(\hat{\mathcal{H}}) \otimes \mathbb{1}_{P\mathcal{H}},$$

$$\Psi \circ \varphi_2(\mathcal{P}(\mathcal{H}_2)) = \mathbb{1}_{\mathcal{H}} \otimes \mathcal{P}(P\mathcal{H}),$$

it is obvious that the compatibility condition (1.2) is fulfilled. Moreover, for any q atom in $\mathcal{P}(\mathcal{H}_1)$, r atom in $\mathcal{P}(\mathcal{H}_2)$ we have

$$(\varphi_1(q) \wedge \varphi_2(r)) = (p_{\varphi(q)} \otimes \mathbb{1}_{P\mathcal{H}}) \wedge (\mathbb{1}_{\hat{\mathcal{H}}} \otimes r) = p_{\varphi(q)} \otimes r$$

which implies $\varphi_1(q) \wedge \varphi_2(r) \in A(\mathcal{P}(\mathcal{H}))$. Hence condition (1.3) is verified.

Finally, it is obvious that $\varphi_1(\mathbb{1}_{\mathcal{H}_1}) = \mathbb{1}_{\mathcal{H}}$, $\varphi_2(\mathbb{1}_{\mathcal{H}_2}) = \mathbb{1}_{\mathcal{H}}$ and $\varphi_1(\mathcal{P}(\mathcal{H}_1)) = \varphi(\mathcal{P}(\tilde{\mathcal{H}})) = \tilde{\mathcal{L}}$. This completes the proof of (2.4) $\Rightarrow$ (2.1).

(2.1) $\Rightarrow$ (2.4). Let $\mathcal{H}_1, \mathcal{H}_2, \varphi_1, \varphi_2$ be Hilbert spaces and maps satisfying conditions (1.1) – (1.4). Define $\mathcal{L}_1, \mathcal{L}_2$ to be:

$$\mathcal{L}_1 = h_1(\mathcal{P}(\mathcal{H}_1)) = \tilde{\mathcal{L}}, \qquad \mathcal{L}_2 = h_2(\mathcal{P}(\mathcal{H}_2)).$$

From (1.2), we infer that $\mathcal{L}_1'' \subset \mathcal{L}_2'$, $\mathcal{L}_2'' \subset \mathcal{L}_1'$.

Suppose now that (2.4) is not true, i.e. that there exist three atoms P, Q, R such that for any $\alpha \in C$: $PQRP \neq \alpha P$. It is easy to see that this compels $\mathcal{L}_{1,P}'' = \{PAP|_{P\mathcal{H}}; A \in \mathcal{L}_1''\}$ to be a non-trivial von Neumann algebra in $P\mathcal{H}$, hence

$$\mathcal{L}_{2,P}'' \subset \mathcal{L}_{1,P}' \subsetneq \mathcal{P}(P\mathcal{H}). \tag{3.1}$$

(We follow Dixmier's notations.)

Since $P \in \mathcal{L}_2'$, we have $\mathcal{L}_{2,P}'' = \{A|_{P\mathcal{H}}; A \in \mathcal{L}_2''\}$, hence $P(\mathcal{L}_{2,P}'') \supset \{Q \wedge P|_{P\mathcal{H}}; Q \in P(\mathcal{L}_2'')\}$. Consider now the map

$$\varphi_{2,P}\colon \mathscr{P}(\mathcal{H}_2) \to \mathscr{P}(P\mathcal{H})$$
$$Q \mapsto \varphi_2(Q) \wedge P|_{P\mathcal{H}}.$$

Since P is compatible with $\mathcal{L}_2$, and since φ_2 is a unitary c-morphism, this $\varphi_{2,P}$ is a unitary c-morphism mapping atoms to atoms (because of (1.3)).

If $\dim \mathcal{H}_2 \geqslant 3$, this compels $\varphi_{2,P}$ to be surjective (see [4], Corollary 4.2); hence, $P(\mathcal{L}''_{2,P}) \supset \varphi_{2,P}(\mathscr{P}(\mathcal{H}_2)) = \mathscr{P}(P\mathcal{H})$. This is impossible, however, because of (3.1). If $\dim \mathcal{H}_2 = 1$, the only atom in $\mathscr{P}(\mathcal{H}_2)$ is $\mathbb{1}_{\mathcal{H}_2}$, which implies that the atoms R in $\mathcal{L}_1$ are one-dimensional (because of (1.1) and (1.3)). In this case, it is impossible that (2.4) could be violated. If $\dim \mathcal{H}_2 = 2$, hence $\dim P\mathcal{H} = 2$ ($\varphi_{2,P}$ is a unitary c-morphism mapping atoms to atoms), then, since $\mathcal{L}''_{1,P}$ is non-trivial, $\mathcal{L}''_{1,P}$ contains one-dimensional projectors. It is easy to check that this implies that $\mathcal{L}'_{1,P}$ contains at most two different one-dimensional projectors, which is impossible, since $\mathcal{L}'_{1,P} \in \mathcal{L}''_{2,P} \in \mathscr{P}(C^2)$.

So in every case our supposition that (2.4) is not true leads to an inconsistency. Hence (2.4) is fulfilled.

(2.3) $\Rightarrow$ (2.4). This was proven in [5]: the proof given there was independent of $\dim \tilde{\mathcal{H}}$.

(2.4) $\not\Rightarrow$ (2.3). A counterexample was given in [5], (counterexample 3.9).

(2.3) $\Rightarrow$ (2.1) and (2.1) $\not\Rightarrow$ (2.3). These two statements are now immediate consequences of (2.4) $\Rightarrow$ (2.1), (2.3) $\Rightarrow$ (2.4) and (2.4) $\not\Rightarrow$ (2.3).

From this theorem, we see that condition (2.4) is somehow a privileged one: it is a necessary and sufficient condition, even in the pathological case $\dim \tilde{\mathcal{H}} = 2$, for the sub-PROP $\tilde{\mathcal{L}}$ to represent a physical subsystem. In the next section, we try to give a physical interpretation for this generally valid criterion (2.4).

4. A PHYSICAL INTERPRETATION OF CONDITION (2.4)

To make a physical interpretation of condition (2.4), we have to introduce some concepts defined in [3]. For more details we refer the reader to [3].

Let $\mathcal{L}$ be the PROP of a physical system S. From the construction of $\mathcal{L}$ in [3] we know that $\mathcal{L}$ is just the collection of all equivalence classes of yes–no experiments which we can perform on S. So if $b \in \mathcal{L}$ then b is the class of all the yes–no experiments which measure the same property of the system S. In general, a yes–no experiment is an experiment which profoundly disturbs the system and which may even destroy the system. It is possible, however, to study a special kind of yes–no experiments, which are called ideal measurements of the first kind (see [3], Chapter 4):

DEFINITION. *A yes–no experiment β is called an ideal measurement of the first kind if:*

(a) *every time the answer "yes" is obtained, one can say that the property b measured by β is true immediately after the measurement (i.e. every yes–no experiment measuring b will give with certainty the answer "yes" after this first measurement),*

(b) *every property which is compatible with b, and which is true immediately before the experiment, is still true after the measurement if the answer was "yes".*

One can show ([3], Theorem 4.3), that if p is the state of S before we perform an ideal measurement of the first kind β, and if the answer is yes, then the state of S immediately after the measurement is $\Phi_b(p) = (p \vee b') \wedge b$. (It is easy to check that, if $p \not\perp b$, this is again an atom, hence a state) Generalizing this, we define a map Φ_b:

$$\begin{aligned} \Phi_b : \mathcal{L} &\to \mathcal{L} \\ c &\mapsto (c \vee b') \wedge b. \end{aligned}$$

In the case where $\mathcal{L} = \mathcal{P}(\mathcal{H})$, this map Φ_G is, in fact, induced by the projection operator P_G corresponding to the closed subspace G of $\mathcal{H}$. Indeed, one can check that

$$\forall H \in \mathcal{P}(\mathcal{H}): \Phi_G(H) = \{P_G(x); x \in H\}.$$

Moreover, $\forall x: \Phi_G(\bar{x}) = \overline{P_G(x)}$.
We return now to condition (2.4): $P_G P_H P_R P_G = \lambda P_G$, where G, H, R are atoms, i.e. maximal measurements in $\mathcal{L}$.
We can rewrite (2.4) as follows: (granted that $G \not\perp H,\ H \not\perp R,\ R \not\perp G$)

$$\overline{P_G(P_H P_R P_G(x))} = \overline{P_G(x)}$$

or

$$\Phi_G(\overline{P_H P_R P_G(x)}) = \Phi_G(\bar{x})$$

or

$$\Phi_G \Phi_H \Phi_R \Phi_G = \Phi_G.$$

Hence (2.4) is equivalent to

$$\Phi_G \Phi_H \Phi_R \Phi_G = \Phi_G \qquad \text{for any } G, H, R \text{ with } G \not\perp H,\ H \not\perp R,\ R \not\perp G. \tag{4.1}$$

This relation can be derived directly, without using von Neumann algebras, from the analysis made in [2]. Indeed, let us assume again the existence of $\mathcal{P}(\mathcal{H}_1)$, $\mathcal{P}(\mathcal{H}_2)$, h_1, h_2 satisfying (1.1) – (1.4), and define, as in [2], maps $u_{\bar{x}_1}$:

$$\begin{aligned} u_{\bar{x}_1} : \mathcal{P}(\mathcal{H}_2) &\to \mathcal{P}(\mathcal{H}) \\ G_2 &\mapsto h_2(G_2) \wedge h_1(\bar{x}_1). \end{aligned}$$

On the other hand, we introduce maps $w_{\bar{x}_1}$:

$$\begin{aligned} w_{\bar{x}_1} : \mathcal{P}(\mathcal{H}) &\to \mathcal{P}(\mathcal{H}_2) \\ G &\mapsto u_{\bar{x}_1}^{-1}(\Phi_{h_1(\bar{x}_1)}(G)). \end{aligned}$$

Using the compatibility condition (1.2), it is easy to see that

$$w_{\bar{x}_1} \circ u_{\bar{y}_1} = 0 \quad \text{if} \quad \bar{x}_1 \perp \bar{y}_1$$
$$\mathbb{1}_{\mathcal{H}_2} \quad \text{otherwise}.$$

On the other hand

$$u_{\bar{x}_1} \circ w_{\bar{x}_1} = \Phi_{h_1(\bar{x}_1)}.$$

So

$$\Phi_{h_1(\bar{x}_1)}\Phi_{h_1(\bar{y}_1)}\Phi_{h_1(\bar{z}_1)}\Phi_{h_1(\bar{x}_1)} = u_{\bar{x}_1} w_{\bar{x}_1} u_{\bar{y}_1} w_{\bar{y}_1} u_{\bar{z}_1} w_{\bar{z}_1} u_{\bar{x}_1} w_{\bar{x}_1}$$

$$= u_{\bar{x}_1} w_{\bar{x}_1} = \Phi_{h_1(\bar{x}_1)} \text{ if } \bar{x}_1 \not\perp \bar{y}_1,\ \bar{y}_1 \not\perp \bar{z}_1,\ \bar{z}_1 \not\perp \bar{x}_1.$$

So the condition for which we will try to give a physical interpretation is (41).

Suppose that there exist ideal measurements of the first kind γ, η, ρ corresponding with the properties G, H, R, respectively. If the system is in a state p, and we perform consecutively the measurements $\gamma, \eta, \rho, \gamma$, getting each time the answer "yes", then finally the state of the system will be $\Phi_G \Phi_H \Phi_R \Phi_G(p) = \Phi_G(p)$. If $\tilde{\mathcal{L}}$ represents a physical subsystem $\tilde{S}$ of S, the measurements γ, η, ρ are, in fact, measurements on $\tilde{S}$ only, and they should have no influence on the other constituent of S. So from this point of view (4.1) is an expression of the fact that S is made up of two independent subsystems: a sequence of ideal measurements of the first kind as given above gives the same results as it would if the other system did not exist. Indeed, condition (4.1) is trivially true for one-dimensional G, H, R; the fact that the sub-PROP $\tilde{\mathcal{L}}$ of $\mathcal{L}$ represents a physical subsystem $\tilde{S}$ of S accounts for its remaining true for S.

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