MEASURES FOR MORE QUADRATIC PATH INTEGRALS*

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ABSTRACT. We show that the coherent state matrix elements of the quantum mechanical propagator for all quadratic Hamiltonians may be represented as the limit of path integrals with respect to appropriately modified Wiener measures as the associated diffusion constant tends to infinity.

In this letter we state some results concerning path integrals with genuine mathematically well-defined measures to define matrix elements of the quantum mechanical evolution operator associated to a Hamiltonian $\mathcal{H}$. These results are a continuation, and in a certain sense also a generalization, of related work published earlier [1, 2, 3]. More specifically, we claim that for all quadratic Hamiltonian with time-dependent linear terms, the matrix elements between coherent states of the evolution operator $T\exp[-i\int_0^T\mathcal{H}(t)\,\mathrm{d}t]$ can be written as a limit of well-defined path integrals involving Wiener measures. Our procedure follows a limiting approach introduced in [3], which is simpler and more intuitive than the projection technique used in [1], and which covers a wider class of Hamiltonians.

Explicitly, our claim is that for

$$\mathcal{H}(t) = \tfrac{1}{2}[\alpha P^2 + \beta Q^2 + \gamma(PQ + QP)] + r(t)Q + s(t)P \tag{1}$$

we have

$$\begin{aligned}\langle p'', q''|\,T\exp\left[-i\int_0^T \mathcal{H}(t)\,\mathrm{d}t\right]|p', q'\rangle\\ = \lim_{\nu\to\infty}\{\exp(\tfrac{1}{2}\nu T)I_{\nu,\mathcal{H}}(p'', q'', T; p', q', o)\},\end{aligned} \tag{2}$$

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where $|p, q\rangle \equiv \exp[i(pQ - qP)]\,|o\rangle$ are the usual canonical coherent states with $(Q + iP)|o\rangle = 0$, and where $I_{\nu,\mathcal{H}}$ is the path integral

$$\begin{aligned} &I_{\nu,\mathcal{H}}(p'', q'', T; p', q', o) \\ &\quad = 2\pi \int \exp\left\{\tfrac{1}{2} i \int (p\,\mathrm{d}q - q\,\mathrm{d}p) - i \int H(p, q, t)\,\mathrm{d}t\right\} \mathrm{d}\mu_\nu(p, q). \end{aligned} \tag{3}$$

Here the function H is defined as

$$\begin{aligned} H(p, q, t) &\equiv \langle p, q\,|\mathcal{H}(t)|\,p, q\rangle \\ &= \tfrac{1}{2}(\alpha p^2 + \beta q^2 + 2\gamma pq) + r(t)q + s(t)p + \tfrac{1}{4}(\alpha + \beta) \end{aligned} \tag{4}$$

and the measure $\mathrm{d}\mu_\nu(p, q)$ is given by

$$\mathrm{d}\mu_\nu(p, q) = \exp\left\{\frac{1}{\nu}\int\left[\frac{\partial H}{\partial p}\,\mathrm{d}q - \frac{\partial H}{\partial q}\,\mathrm{d}p\right] - \frac{1}{2\nu}\int\left[\left(\frac{\partial H}{\partial p}\right)^2 + \left(\frac{\partial H}{\partial q}\right)^2\right]\mathrm{d}t\right\} \mathrm{d}\mu_w(p)\,\mathrm{d}\mu_w(q), \tag{5}$$

where $\mathrm{d}\mu_w(x)$, $x \equiv p$ or q, is a Wiener measure pinned so that $x(T) = x''$, $x(0) = x'$, and normalized such that

$$\int \mathrm{d}\mu_w(x) = (2\Pi\nu T)^{-1/2} \exp\left[-(x'' - x')^2/2\nu T\right]. \tag{6}$$

For any H it follows from (5) that $\int \mathrm{d}p''\,\mathrm{d}q'' \int \mathrm{d}\mu_\nu(p, q) = 1$, a fact clear from the formal expression

$$\mathrm{d}\mu_\nu(p, q) = \mathcal{N} \exp\left\{-\frac{1}{2\nu}\int\left[\left(\dot p + \frac{\partial H}{\partial p}\right)^2 + \left(\dot q - \frac{\partial H}{\partial p}\right)^2\right]\mathrm{d}t\right\} \prod_t \mathrm{d}p(t)\,\mathrm{d}q(t) \tag{7}$$

along with the customary definition of such expressions in the manner of Itô [4]; as usual the coefficient $\mathcal{N}$ in (7) is a formal (divergent) H-independent normalization factor chosen so as to ensure the Wiener measure normalization (6).

Note how the $\dot p^2$, $\dot q^2$-terms in the formal expression (7) that lead to the Wiener measures $\mathrm{d}\mu_w(p)$, $\mathrm{d}\mu_w(q)$ in (5), arise from terms resembling the classical Hamilton equations for the Hamiltonian function $H(p, q)$. The picture behind the definition (3) of $I_{\nu,\mathcal{H}}$ is therefore one of a contribution given by a classical path [minimizing the exponential in (7)], and paths lying 'near' to it, where 'nearness' is measured by ν, which plays the role of a diffusion constant. This means that in the limit, as ν tends to infinity, more and more paths become 'relevant' in the evaluation of $I_{\nu,\mathcal{H}}$. In fact, in the limit $\nu \to \infty$ one could *formally* say that the term $(1/2\nu)\int[(\dot p + \partial H/\partial q)^2 + (\dot q - \partial H/\partial p)^2]\,\mathrm{d}t$ just 'disappears' from (7), yielding an expression for the right-hand side of (2) given by

$$\mathcal{N}' \int \exp\left\{\tfrac{1}{2} i \int (p\,\mathrm{d}q - q\,\mathrm{d}p) - i \int H(p, q, t)\,\mathrm{d}t\right\} \prod_t \mathrm{d}p(t)\,\mathrm{d}q(t) \tag{8}$$

where we have absorbed the factor $\exp(\nu T/2)$ into $\mathcal{N}$ to give $\mathcal{N}'$. But (8) is just the formal expression for the path integral one obtains by applying the usual arguments leading to path integrals in a coherent state setting (see [5]) for quite general Hamiltonians. It is therefore conceivable that the prescription (2) + (3) + (5) might also work for nonquadratic Hamiltonians.

The evaluation of $I_{\nu,\mathcal{H}}$ was carried out by a standard procedure. As a Gaussian path integral, its value is given exactly by the stationary phase approximation; i.e., (3) can be evaluated by taking the value of the exponential [including the exponential factor contained in $d\mu_\nu$, as given by (7)] at the path extremising the exponent, and multiplying this value by an appropriate numerical constant. To determine this constant, we used a differential equation associated to the path integral (3). The result is

$$
\begin{aligned}
I_{\nu,\mathcal{H}} = [-\det M_\nu]^{1/2}\ \exp[\tfrac{1}{4} i(\alpha+\beta)T]\,\mathcal{I}_{\mathcal{H}}(p'', q'', T; p', q', o)\,\times \\
\times\ \exp\{\tfrac{1}{2} i[M_{\nu,11}\tilde{p}^2 + M_{\nu,22}\tilde{q}^2 + (M_{\nu,12}+M_{\nu,21})\tilde{p}\tilde{q}]\}
\end{aligned}
\tag{9}
$$

with

$$
\begin{aligned}
\mathcal{I}_{\mathcal{H}}(p'', q'', T; p', q', o) = \exp\Big(&-\tfrac{1}{2} i[p''\mathfrak{Q}(T;p',q') - q''\mathfrak{P}(T;p'q')] + \\
&+\tfrac{1}{2} i\int_0^T dt\{r(t)\mathfrak{Q}(t;p',q') + s(t)\mathfrak{P}(t;p',q') - \\
&-p''\mathfrak{Q}[T-t; r(t), -s(t)] + q''\mathfrak{P}[T-t; r(t), -s(t)]\} + \\
&+\tfrac{1}{2} i\int_0^T dt_1 \int_0^{t_1} dt_2\,\{r(t_1)\mathfrak{Q}[t_1-t_2; r(t_2), -s(t_2)] + \\
&+ s(t_1)\mathfrak{P}[t_1-t_2; r(t_2), -s(t_2)]\}\Big).
\end{aligned}
\tag{10}
$$

We have denoted by $\mathfrak{P}(t;p,q)$, $\mathfrak{Q}(t;p,q)$ the solutions, at time t and starting from the initial conditions $\mathfrak{P}(o;p,q) = p$, $\mathfrak{Q}(o;p,q) = q$, of the Hamilton equations associated to the purely quadratic part of the Hamiltonian function H:

$$
\begin{cases}
\dot{\mathfrak{P}} = -\gamma\mathfrak{P} - \beta\mathfrak{Q} \\
\dot{\mathfrak{Q}} = \alpha\mathfrak{P} + \gamma\mathfrak{Q}.
\end{cases}
$$

Introducing the matrices

$$
A = \begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
$$

We can also write

$$\begin{pmatrix}\mathfrak{P}(t;p,q)\\ \mathfrak{Q}(t;p,q)\end{pmatrix}=\exp[JAt]\begin{pmatrix}p\\ q\end{pmatrix}.$$

The symbol M_ν in (9) stands for the matrix

$$M_\nu=[1-e^{JAT}\,e^{-J(A+i\nu)T}]^{-1}, \tag{11}$$

and $M_{\nu,ij}$ are its i,j matrix elements. Finally, we have denoted by $\widetilde{p},\widetilde{q}$ the following expressions

$$\begin{aligned}\widetilde{p}&=p''-\mathfrak{P}(T;p',q')-\int_0^T \mathrm{d}t\,\mathfrak{P}[T-t;r(t),-s(t)]\,,\\ \widetilde{q}&=q''-\mathfrak{Q}(T;p',q')-\int_0^T \mathrm{d}t\,\mathfrak{Q}[T-t;r(t)\;-s(t)]\,.\end{aligned} \tag{12}$$

Note that $\mathscr{I}_{\mathcal{H}}$ and $\widetilde{p},\widetilde{q}$, as defined by (10) and (12) respectively, are completely independent of ν: the ν-dependence of $I_{\nu,\mathcal{H}}$ is completely confined to the matrix M_ν given by (11). To evaluate the limit of $\exp(\nu T/2)I_{\nu,\mathcal{H}}$, we therefore only have to control the limiting behavior of M_ν. It turns out that

$$M_\infty\equiv\lim_{\nu\to\infty}M_\nu=[1+e^{-AJT}\,e^{JAT}]^{-1}(1+iJ). \tag{13}$$

The factor $(1+iJ)$ in M_∞ has determinant zero, which implies $\det M_\infty=0$ as well; this is due to the fact that the presence of the term $-i\nu JT$ in one of the exponentials in (11) causes one of the eigenvalues of M_ν^{-1} to diverge exponentially in ν as $\nu\to\infty$. This limiting behavior of $(-\det M_\nu)^{1/2}$, tending exponentially to zero as $\nu\to\infty$ is, however, exactly counter-balanced by the exponential factor $\exp(\nu T/2)$ in (2). It turns out that

$$\begin{aligned}m_\infty&\equiv\lim_{\nu\to\infty}\;[\exp(\nu T/2)(-\det M_\nu)^{1/2}]\;\exp\;[i(\alpha+\beta)T/4]\\ &=2\{\det\;[(1+iJ)+e^{JAT}(1-iJ)]\}^{-1/2}.\end{aligned} \tag{14}$$

So finally, substituting (13) and (14) into (10), we have

$$\begin{aligned}L&\equiv\lim_{\nu\to\infty}\;[\exp(\tfrac{1}{2}\nu T)I_{\nu,\mathcal{H}}(p'',q'',T;p',q',o)]\\ &=m_\infty\mathscr{I}_{\mathcal{H}}(p'',q'',T;p',q',o)\;\exp\{\tfrac{1}{2}i[M_{\infty,11}\widetilde{p}^2+M_{\infty,22}\widetilde{q}^2+(M_{\infty,12}+M_{\infty,21})\widetilde{p}\widetilde{q}]\}.\end{aligned} \tag{15}$$

As an intermediate step in reducing this to the left-hand side of (2), one can check (using results on coherent state matrix elements of the unitary operators representing canonical transformations; see [6]) that this is exactly the matrix element

$$L = \exp\left(\tfrac{1}{2} i \int_0^T \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \{ r(t_1)\, \mathfrak{Q}[t_1 - t_2 ; r(t_2), -s(t_2)] \quad + \right.$$

$$\left. + s(t_1)\, \mathfrak{P}[t_1 - t_2 ; r(t_2), -s(t_2)] \} \right) \times \tag{16}$$

$$\times \langle\, p'' - \int_0^T \mathrm{d}t\, \mathfrak{P}[T - t; r(t), -s(t)] \,, q'' - \int_0^T \mathrm{d}t\, \mathfrak{Q}[T - t; r(t), -s(t)] \times$$

$$\times \,| \exp[-iT\mathcal{H}_0]\, | p' q' \rangle \,,$$

where $\mathcal{H}_0 = \frac{1}{2}[\alpha P^2 + \beta Q^2 + \gamma(PQ + QP)]$ is the purely quadratic part of $\mathcal{H}$. By differentiation one then shows that

$$L = \langle p'', p'' | T \exp\left(-i \int_0^T \mathrm{d}t \{ \tfrac{1}{2}[\alpha P^2 + \beta Q^2 + \gamma(PQ + QP)] \; + \right.$$

$$\left. + r(t) Q + s(t) P \} \right) | p', q' \rangle$$

thereby establishing our claim (2).

Finally, we want to conclude this letter with some remarks.

(1) *Extension to higher dimensions.* For Hamiltonians of the type $\mathcal{H}(P, Q) = \frac{1}{2} \Sigma_j 1/m_j P_j^2 + V(Q_j) +$ (linear terms in P, Q), where V is quadratic in the Q_j, our result can be trivially extended.

(2) *Time-dependent quadratic part of* $\mathcal{H}$. Our proof can also handle the case where the quadratic part of the Hamiltonian is multiplied by an overall time-dependent coefficient $\lambda(t)$:

$$\mathcal{H}(t) = \tfrac{1}{2}\lambda(t)[\alpha P^2 + \beta Q^2 + \gamma(PQ + QP)] + r(t) Q + s(t) P,$$

where λ is piecewise differentiable. Even the case where α, β, γ are piecewise constant functions of t can still be handled by our proof.

(3) *Classical limit.* Finally it is amusing to note that if instead of taking the limit $\nu \to \infty$, as in (2), we take the limit $\nu \to 0$, we find

$$\lim_{\nu \to 0} I_{\nu, \mathcal{H}} = 2\pi\, \delta[p'' - \mathfrak{P}_f(T; p', q')]\, \delta[q'' - \mathfrak{Q}_f(T; p', q')]$$

where $\mathfrak{P}_f$, $\mathfrak{Q}_f$ denote now the classical evolution under the *full* Hamiltonian function $H(p, q, t)$ [as defined in (4)]. Thus, in a sense ν interpolates between quantum and classical expressions for the propagator.

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