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Measures for Path Integrals

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By exploitation of the overcompleteness of coherent states expressions are presented for path integrals in terms of genuine (Wiener) path-space measures for driven harmonic oscillators which when projected onto the subspace spanned by coherent-state matrix elements yield the appropriate quantum mechanical propagator.

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Path integrals for quantum mechanics have resisted formulations in which genuine measures—real or complex—on q-space and/or p-space paths are involved. This situation is in marked contrast to the Euclidean formulation of quantum mechanics (generalized diffusion theory) in which one has the well-known Feynman-Kac formula based on the Wiener measure that governs q-space Brownian-motion paths.[1] Since the formal "measures" of quantum mechanical path integrals are not countably additive it has been basically necessary to resort to limiting procedures (e.g., lattice-space formulations and subsequent limits) to define the path integral. In this Letter we indicate, for a certain class of dynamical systems and in a projection sense made clear below, that quantum mechanical path integrals involving genuine measures do in fact exist.

The basic idea in our approach is to exploit the overcompleteness inherent in the usual coherent states. Linear dependences among the coherent states imply ambiguities in the integral kernels that represent operators, or, stated otherwise, that an equivalence class of kernels corresponds to the same operator. In this Letter we observe that within the equivalence class corresponding to the quantum mechanical evolution operator for driven harmonic oscillators there exist integral kernels that admit representations as genuine Wiener integrals over phase space. Other dynamical systems formally follow by integration or equivalently by functional differentiation involving the driving function. After outlining the well-founded mathematical formulation of our quantum mechanical path integral, we also give a formal statement of our main result.

Coherent-state properties.—Coherent states are typically defined as ($\hbar \equiv 1$)

$$|p, q\rangle \equiv e^{i(pQ-qP)}|0\rangle, \tag{1}$$

for all real p and q, in terms of canonical Heisenberg operators Q and P, and the normalized ground state $|0\rangle$ of a harmonic oscillator of unit angular frequency. Such states are complete in the sense that the unit operator I admits the resolution

$$I = \int |p, q\rangle\langle p, q| \, (dp\, dq/2\pi) \tag{2}$$

when integrated over all phase space; in fact they are overcomplete, the nonvanishing overlap of two such states,

$$\langle p_2, q_2 | p_1, q_1\rangle = \exp\{\tfrac{1}{2}i(q_2p_1 - p_2q_1) - \tfrac{1}{4}[(p_2-p_1)^2 + (q_2-q_1)^2]\}, \tag{3}$$

being a reflection of that overcompleteness.[2]

Let B be an arbitrary, bounded (for convenience) operator, and consider the expression

$$\langle p_2, q_2 | B | p_1, q_1\rangle = \int\int \langle p_2, q_2 | p'', q''\rangle \mathcal{K}_B(p'', q''; p', q')\langle p', q' | p_1, q_1\rangle (dp''\,dq''/2\pi)(dp'\,dq'/2\pi). \tag{4}$$

Here $\mathcal{K}_B$ is an integral kernel that leads to the left-hand side for arbitrary coherent states. One suitable integral kernel is always given by

$$\mathcal{K}_B(p'', q''; p', q') = \langle p'', q'' | B | p', q'\rangle, \tag{5}$$

but because of the overcompleteness of the coherent states there are infinitely many other linearly independent integral kernels $\mathcal{K}_B$ that also fulfill (4) for a given operator B.[2] All such integral kernels $\mathcal{K}_B$ lie in an equivalence class labeled by the operator B. A generic integral kernel in this equivalence class is conveniently denoted by $\langle p'', q'' | B | p', q'\rangle_{\rm EC}$.

Driven harmonic oscillator.—Let

$$\mathcal{H}(t) = \tfrac{1}{2}\alpha(P^2 + Q^2 - 1) + s(t)Q, \tag{6}$$

which for $\alpha = 1$ is a harmonic-oscillator Hamiltonian with unit angular frequency driven by an external c-number function s. The evolution operator for such a dynamical system is $U = T\exp[-i\int \mathcal{H}(t)dt]$, where T is the usual time-ordering operator. Here and in all expressions that follow, all unspecified time integrations extend from t' to t'' where $-\infty < t' < t'' < \infty$. If we choose $B = U$, then the expression

$$\langle p'', q'', t'' | p', q', t'\rangle \equiv \langle p'', q'' | T\exp[-i\int \mathcal{H}(t)dt] | p', q'\rangle \tag{7}$$

represents the propagator expressed in terms of coherent-state matrix elements. This expression has the customary formal path-integral representation[3] given for the Hamiltonian (6) by

$$\langle p'', q'', t'' | p', q', t'\rangle = \mathfrak{N}\int \exp(i\int\{\tfrac{1}{2}[p(t)\dot q(t) - q(t)\dot p(t)] - \tfrac{1}{2}\alpha[p^2(t) + q^2(t)] - s(t)q(t)\}dt)\textstyle\prod_t dp(t)dq(t), \tag{8}$$

where $\mathfrak{N}$ is a formal normalization constant.

The study of the Hamiltonian (6) and its formal path integral (8) can serve as the starting point for alternative dynamical systems. As one example, we may simply choose $\alpha = 0$. As another example, let $p(t) \to \omega^{-1/2}p(t)$, $q(t) \to \omega^{1/2}q(t)$, $s(t) \to \omega^{-1/2}(t)$, and choose $\alpha = \omega$; then (8) represents the propagator for a driven oscillator of angular frequency ω. Note for the coherent-state path-integral formulation that the formal measure in (8) is invariant under such a transformation.[3] Additionally, as a standard trick, we may integrate (8) over some distribution of s paths thereby converting (8) into an expression for a more general potential, at least in a formal manner.

Equivalence-class propagator.—We now assert that we can write an element of the equivalence class of the evolution operator for the Hamiltonian (6) in the following form[4]:

$$\langle p'', q'', t'' | p', q', t'\rangle_{\rm EC} = 2\pi\exp[\tfrac{1}{2}(t''-t')]\int \exp(\int\{(\tfrac{1}{2}i + \alpha)[p(t)\dot q(t) - q(t)\dot p(t)] - \tfrac{1}{2}\alpha(i+\alpha)[p^2(t) + q^2(t)]$$
$$- s(t)[(i+\alpha)q(t) + \dot p(t)] - \tfrac{1}{2}s^2(t)\}dt)\,d\mu_w^{p'',p'}(p)\,d\mu_w^{q'',q'}(q). \tag{9}$$

In this expression $\mu_w^{x'',x'}$, where either $x \equiv p$ or $x \equiv q$, is a standard Wiener measure on continuous paths pinned so that at t', $x(t') = x'$, and at t'', $x(t'') = x''$. In units where $t' = 0$ and $t'' = T$ then for $0 \le t \le T$, $\langle x(t)\rangle_w = T^{-1}[tx'' + (T-t)x']$, while for $0 \le u \le v \le T$ the connected (c) covariance function reads $\langle x(v)x(u)\rangle_w^c = T^{-1}u(T-v)$, where $\langle(\cdots)\rangle_w \equiv \int(\cdots)d\mu_w^{x'',x'}(x)/\int d\mu_w^{x'',x'}(x)$ and $\int d\mu_w^{x'',x'}(x) = (2\pi T)^{-1/2} \times\exp[-\tfrac{1}{2}(x''-x')^2\text{-}T]$. Since the Wiener measure is a Gaussian measure all correlation functions follow from the mean and covariance.[5]

While (9) has been written in physicists' notation it is in fact mathematically well defined. To see this we need only interpret the terms $(p\dot q - q\dot p)dt$ and $s\dot p\,dt$ in the fashion $p\,dq - q\,dp$ and $s dp$, respectively. As such all the terms in the indicated integrand in (9) are well-defined stochastic integrals, this being the case even if s itself is an element of a stochastic process for which the expectation of $\int s^2(t)dt$

is finite.[6] Since p and q (and s too if so regarded) are independent stochastic variables then all prescriptions (Itô, Stratonovich, etc.) for defining the stochastic integrals are equivalent.[7] Consequently, (9) represents a mathematically well-defined path integral with a genuine measure for an element of the equivalence class of the quantum mechanical propagator.

The evaluation of (9) is straightforward and is given by

$$\begin{aligned}\langle p'', q'', t'' | p', q', t'\rangle_{EC} = A \exp\bigl(&\tfrac{1}{2}i(q'' p_\alpha' - p'' q_\alpha') \\ &- \tfrac{1}{4}B\{[q'' - q_\alpha' + \textstyle\int \sin\alpha(t'' - t)s(t)dt]^2 + [p'' - p_\alpha' + \textstyle\int \cos\alpha(t'' - t)s(t)dt]^2\} \\ &- \tfrac{1}{2}i \textstyle\int [\cos\alpha(t'' - t)q'' + \cos\alpha(t - t')q' - \sin\alpha(t'' - t)p'' + \sin\alpha(t - t')p'] s(t)dt \\ &+ \tfrac{1}{4}i \textstyle\iint dt_2\, dt_1 \sin\alpha|t_2 - t_1| s(t_2)s(t_1)\bigr), \end{aligned} \tag{10}$$

where $q_\alpha' \equiv q'\cos\alpha(t'' - t') + p'\sin\alpha(t'' - t')$, $p_\alpha' \equiv -q'\sin\alpha(t'' - t') + p'\cos\alpha(t'' - t')$, and

$$A = \{1 - \exp[-(t'' - t')]\}^{-1}, \quad B = \coth[\tfrac{1}{2}(t'' - t')]. \tag{11}$$

When the kernel (10) is inserted into (4) the result is an expression for the correct propagator (7), where the functional form of the correct propagator is also given by (10), the only change being that $A = B = 1$. Observe then from (11) that as the time interval $t'' - t'$ becomes arbitrarily large the equivalence-class propagator (10) actually converges to the correct propagator without any need for projection via Eq. (4)!

The preceding analysis holds for any value of α, but now let us assume that $\alpha \neq 0$ and for convenience that $\alpha = 1$. We can then recast (9) into an alternative, equivalent expression by incorporating the factor $\frac{1}{2}(p^2 + q^2)$ into the measure, changing the standard Wiener measure to an associated Ornstein-Uhlenbeck measure.[1] Thus (9) may be rewritten as

$$\begin{aligned}\langle p'', q'', t'' | p', q', t'\rangle_{EC} = 2\pi \exp[\tfrac{1}{2}(t'' - t')] \textstyle\int \exp\bigl(\int \{(\tfrac{1}{2}i + 1)[p(t)\dot{q}(t) - q(t)\dot{p}(t)] - \tfrac{1}{2}i[p^2(t) + q^2(t)] \\ - s(t)[(i+1)q(t) + \dot{p}(t)] - \tfrac{1}{2}s^2(t)\}dt\bigr) d\mu_{OU}{}^{p'', p'}(p)\, d\mu_{OU}{}^{q'', q'}(q). \end{aligned} \tag{12}$$

Here $\mu_{OU}{}^{x'', x'}$, where either $x \equiv p$ or $x \equiv q$, is an Ornstein-Uhlenbeck measure on continuous paths pinned so that at t', $x(t') = x'$, and at t'', $x(t'') = x''$. In units where $t' = 0$ and $t'' = T$ then for $0 \leq t \leq T$

$$\langle x(t)\rangle_{OU} = (1 - e^{-2T})^{-1}[e^{-(T-t)}(1 - e^{-2t})x'' + e^{-t}(1 - e^{-2(T-t)})x'], \tag{13}$$

while for $0 \leq u \leq v \leq T$ the connected covariance function reads

$$\langle x(v)x(u)\rangle_{OU}{}^c = \tfrac{1}{2}e^{-(v-u)} - \tfrac{1}{2}(1 - e^{-2T})^{-1}(e^{-v-u} + e^{-2T+v+u} - e^{-2T+u-u} - e^{-2T+u-v}), \tag{14}$$

where $\langle(\cdots)\rangle_{OU} \equiv \int(\cdots)d\mu_{OU}{}^{x'', x'}(x) / \int d\mu_{OU}{}^{x'', x'}(x)$ and

$$\int d\mu_{OU}{}^{x'', x'}(x) = \pi^{-1/2}(1 - e^{-2T})^{-1/2} e^{-T/2} \exp\left[-(1 - e^{-2T})^{-1}\left(\frac{1 + e^{-2T}}{2}(x'^2 + x''^2) - 2x'x''e^{-T}\right)\right]. \tag{15}$$

Since the Ornstein-Uhlenbeck measure is a Gaussian measure all correlation functions follow from (13) and (14). For $T = t'' - t' \ll 1$ these relations reduce to those given above for the Wiener measure.[8]

If we specialize to $p'' = q'' = p' = q' = 0$ and pass to the limit $t'' \to \infty$ and $t' \to -\infty$ then it follows from (11) and (12), for all square-integrable s, that

$$\begin{aligned}&\langle 0| T\exp\{-i\textstyle\int_{-\infty}^{\infty} [\tfrac{1}{2}(P^2 + Q^2 - 1) + s(t)Q]dt\}|0\rangle \\ &= \langle 0|T\exp[-i\textstyle\int_{-\infty}^{\infty} s(t)Q(t)dt]|0\rangle \\ &= \lim_{\substack{t''\to\infty \\ t'\to-\infty}} 2\pi\exp\{-\tfrac{1}{2}\textstyle\int[s^2(t) - 1]dt\}\int \exp\bigl(\{(\tfrac{1}{2}i + 1)[p(t)\dot{q}(t) - q(t)\dot{p}(t)] - \tfrac{1}{2}i[p^2(t) + q^2(t)] \\ &\qquad - s(t)[(i+1)q(t) + \dot{p}(t)]\}dt\bigr) d\mu_{OU}{}^{0,0}(p)\, d\mu_{OU}{}^{0,0}(q). \end{aligned} \tag{16}$$

In this expression the second form corresponds to the interaction representation where $Q(t) \equiv Q\cos t + P\sin t$. Since the time interval has diverged the limit of path integrals in (16) actually converges to the desired ground-state expectation value, the evaluation of which is given by the well-known expression $\exp[-\frac{1}{4}\int_{-\infty}^{\infty}\int s(v)s(u)e^{-i|v-u|}dv\,du]$.

Formal statement.—In order to gain insight and help interpret the integrand in our main result (9), it is instructive to recast this expression in a formal manner as [cf. (8)]

$$\langle p'', q'', t''|p', q', t'\rangle_{\mathrm{EC}} = \mathfrak{N}\iiint e^{iI}\exp\left\{-\frac{1}{2}\int\left[\left(\dot p+\frac{\partial H}{\partial q}\right)^2+\left(\dot q-\frac{\partial H}{\partial p}\right)^2\right]dt\right\}\Pi_t\, dp(t)dq(t), \tag{17}$$

where $\mathfrak{N}$ is a formal normalization constant, and

$$I \equiv \int\{\tfrac{1}{2}[p(t)\dot q(t) - q(t)\dot p(t)] - H(p(t), q(t), t)\}\, dt, \tag{18}$$

$$H \equiv H(p(t), q(t), t) \equiv \tfrac{1}{2}\alpha[p^2(t)+q^2(t)] + s(t)q(t). \tag{19}$$

Thus we see that the phase of the integrand is given as usual by the classical action, while the drift terms in the measure are those dictated by the classical equations of motion. Note that the total weight of the measure, integrated over the final values as well, is independent of the classical Hamiltonian H, implying that the measure just redistributes the weight when $H=0$ without changing its total value. It is interesting to speculate if (17) may hold true for more general Hamiltonians than those given in (19).

We emphasize that by exploiting the overcompleteness of the coherent states we have been able to formulate quantum mechanical path integrals as integrals involving genuine (countably additive) measures. The principal price needed to accomplish this result is the appearance of a different "action" expression than the usual one. Our method has been illustrated for driven harmonic oscillators which can be extended by integration over the external source to more general potentials. Apparently the present formulation cannot rigorously lead to genuine measures for general local potentials when integrated over external sources, but it is hoped that this aspect can be rectified by a more elaborate analysis of the type outlined here that exploits other elements of the equivalence class of the evolution operator. A detailed discussion and derivation of the path-integral expressions given in this Letter will be presented elsewhere.[9]

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[1]See, e.g., B. Simon, *Functional Integrals and Quantum Physics* (Academic, New York, 1979).

[2]See, e.g., J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (Benjamin, New York, 1968), Chap. 7.

[3]See, e.g., J. R. Klauder, Acta Phys. Austriaca, Suppl. XXII, 3 (1980), and in *Path Integrals*, edited by G. J. Papadopoulos and J. T. Devreese (Plenum, New York, 1978), p. 5.

[4]The validity of (9) may be verified by direct computation since only Gaussian integrals are involved. A detailed derivation will be presented elsewhere (I. Daubchies and J. R. Klauder, to be published).

[5]In units where $t'=0$ and $t''=T$ the measure $\mu_w^{x'',x'}$ is related to the standard Wiener probability measure $\nu_w^{x'}$ pinned at $t=0$ so that $x(0)=x'$ according to

$$\int\delta(x(T)-x'')(\cdots)d\nu_w^{x'}(x) = \int(\cdots)d\mu_w^{x'',x'}(x).$$

The probability measure $\nu_w^{x'}$ has mean x' and a connected covariance given [in the notation of the text] by u. These Wiener measures are just the ones that enter the Feynman-Kac formula for a particle of unit mass [cf. Ref. 1].

[6]More generally, (9) is well defined if $\int s^2(t)dt$ is finite with probability 1. See, e.g., H. P. McKean, Stochastic *Integrals* (Academic, New York, 1969).

[7]See, e.g., K. Itô, in *Mathematical Problems in Theoretical Physics*, edited by G. Dell-Antonio, S. Doplicher, and R. Jona-Lasinio (Springer-Verlag, New York, 1975), p. 218.

[8]In units where $t'=0$ and $t''=T$ the measure $\mu_{\mathrm{OU}}^{x'',x'}$ is related to the stationary Ornstein-Uhlenbeck probability measure ν_{OU} according to

$$\int\delta(x(T)-x'')\delta(x(0)-x')(\cdots)d\nu_{\mathrm{OU}}(x) = \pi^{-1/2}e^{-T/2}\exp(\tfrac{1}{2}(x'^2+x''^2))\int(\cdots)d\mu_{\mathrm{OU}}^{x'',x'}(x).$$

The stationary probability measure ν_{OU} has mean zero and covariance given by just the first term in (14).

[9]Daubchies and Klauder, Ref. 4.