WAVELETS ON THE INTERVAL

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The construction of orthonormal wavelet bases or of pairs of dual, biorthogonal wavelet bases for $L^2(\mathbb{R})$ is now well understood. For the construction of orthonormal bases of compactly supported wavelets for $L^2(\mathbb{R})$, in particular, one starts with a trigonometric polynomial $m_0(\xi) = \sum_n c_n e^{-in\xi}$, satisfying $m_0(0) = 1$ and $|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1$, as well as some mild technical conditions. The corresponding scaling function ϕ and wavelet ψ are defined by $\hat{\phi}(\xi) = (2\pi)^{-1/2} \prod_{j=1}^{\infty} m_0(2^{-j}\xi)$ and $\hat{\psi}(\xi) = e^{-i\xi/2}\, \overline{m_0(\xi/2+\pi)}\, \hat{\phi}(\xi/2)$. The functions $\psi_{j,k}(x) = 2^{-j/2}\ \psi(2^{-j}x - k)$, $j, k \in \mathbb{Z}$, then constitute an orthonormal basis for $L^2(\mathbb{R})$. For fixed $j \in \mathbb{Z}$, the $\phi_{j,k}(x) = 2^{-j/2}\ \phi(2^{-j}x - k)$ are an orthonormal basis for a subspace $V_j \subset L^2(\mathbb{R})$; the spaces V_j constitute a multiresolution analysis, meaning in particular that $\ldots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset \ldots$, with $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$, $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$ and $\mathrm{Proj}_{V_{j-1}} f = \mathrm{Proj}_{V_j} f + \sum_{k\in\mathbb{Z}} \langle f,\ \psi_{j,k}\rangle\ \psi_{j,k}$. (See Mallat[2] (1989), Meyer[3] (1990) or Daubechies[4,6] (1988,1992) for more details.) Smoothness for ψ implies that m_0 has to have a zero at π of sufficiently high multiplicity. More precisely,

$$\psi \in C^k(\mathbb{R}) \Longrightarrow \int dx\ x^\ell\ \psi(x) = 0 \quad \ell = 0, \ldots, k \Longleftrightarrow \frac{d^\ell}{d\xi^\ell}\, m_0 \Big|_{\xi=\pi} = 0 \quad \ell = 0, \ldots, k\,.$$

This in turn implies that m_0 has at least $2k$ non-zero coefficients.

By far the oldest example of such an orthonormal basis of compactly supported wavelets is the Haar basis, with $m_0(\xi) = \frac{1}{2}(1 + e^{-i\xi})$. Other examples, with arbitrarily high smoothness, were constructed in Daubechies[4] (1988). They correspond to m_0 of the type $m_0(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^N Q_N(\xi)$, where $Q_N(\xi)$ is a polynomial of order $N-1$ in $e^{-i\xi}$. The resulting ϕ and ψ have support width $2N-1$; their degree of smoothness increases linearly with N.

These smoother wavelets provide not only orthonormal bases for $L^2(\mathbb{R})$, but also inconditional bases for function spaces consisting of more regular functions. In particular (Meyer[3] (1990)), if $\psi \in C^r(\mathbb{R})$, then the $\phi_{0,k}$, $k \in \mathbb{Z}$ and $\psi_{-j,k}$, $j \in \mathbb{N}$, $k \in \mathbb{Z}$, provide an unconditional basis for the function spaces $C^s(\mathbb{R})$, for all $s < r$. The reason why wavelet bases (unlike

Fourier series) can provide unconditional bases for C^r-spaces is essentially that the wavelets ψ have vanishing moments. Imposing such vanishing moments is equivalent to requiring that any polynomial of degree less than or equal to $N-1$ can be written as a linear combination of the $\phi(x-n)$.

Except for the Haar basis, the basic wavelet in an orthonormal basis of compactly supported wavelets cannot have a symmetry or antisymmetry axis. Symmetry can be recovered, without giving up the compact support, if the orthogonality requirement is relaxed. In that case one builds two different (but related) multiresolution hierarchies of spaces, $\ldots V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset \ldots$ and $\ldots \tilde{V}_2 \subset \tilde{V}_1 \subset \tilde{V}_0 \subset \tilde{V}_{-1} \subset \tilde{V}_{-2} \subset \ldots$, corresponding to two scaling functions ϕ and $\tilde{\phi}$ and two wavelets ψ and $\tilde{\psi}$. They are defined by means of two trigonometric polynomials m_0 and $\tilde{m}_0$, satisfying $m_0(\xi)\,\overline{\tilde{m}_0(\xi)} + m_0(\xi+\pi)\,\overline{\tilde{m}_0(\xi+\pi)} = 1$; we have then $\hat{\phi}(\xi) = (2\pi)^{-1/2}\prod_{j=1}^{\infty} m_0(2^{-j}\xi)$, $\hat{\tilde{\phi}}(\xi) = (2\pi)^{-1/2}\prod_{j=1}^{\infty} \tilde{m}_0(2^{-j}\xi)$, $\hat{\psi}(\xi) = e^{-i\xi/2}\,\overline{\tilde{m}_0(\xi/2+\pi)}\,\hat{\phi}(\xi/2)$, $\hat{\tilde{\psi}}(\xi) = e^{-i\xi/2}\,\overline{m_0(\xi/2+\pi)}\,\hat{\tilde{\phi}}(\xi/2)$. Under some extra technical conditions the $\psi_{j,k}$ and the $\tilde{\psi}_{j,k}$ constitute dual Riesz bases for $L^2(\mathbb{R})$, i.e. $\langle \psi_{j,k}, \tilde{\psi}_{j',k'} \rangle = \delta_{j,j'}\delta_{k,k'}$. For proofs and examples, see Cohen, Daubechies and Feauveau[7] (1992). There exist two possibilities leading to symmetry for ψ, $\tilde{\psi}$: if $m_0, \tilde{m}_0$ have an even number of coefficients, then $\phi(x)$ is symmetric, and ψ is antisymmetric around $x = 1/2$; if m_0 and $\tilde{m}_0$ have an odd number of coefficients, then ϕ and ψ are both symmetric, $\phi(x)$ around $x = 0$, $\psi(x)$ around $x = 1/2$. Smoothness for these "biorthogonal" wavelet bases again requires vanishing moments; we have now

$$\psi \in C^k(\mathbb{R}) \Longrightarrow \int dx\, x^\ell\, \tilde{\psi}(x) = 0 \quad \ell = 0, \ldots, k \Longleftrightarrow \frac{d^\ell}{d\xi^\ell}\, m_0\Big|_{\xi=\pi} = 0 \quad \ell = 0, \ldots, k\,.$$

All the above concerns bases for $L^2(\mathbb{R})$. In many applications, however, one is interested in problems confined to an interval rather than the whole line. Examples are numerical analysis (with boundary conditions at the edges of the interval), or image analysis (where the domain of interest is the cartesian product of two intervals). Let us assume that the interval is $[0,1]$. It is very easy to restrict the Haar basis for $L^2(\mathbb{R})$ to a basis for $L^2([0,1])$; starting from the collection $\{\phi_{0,k};\ k \in \mathbb{Z}\} \cup \{\psi_{j,k};\ j \le 0, k \in \mathbb{Z}\}$, which is an orthonormal basis for $L^2(\mathbb{R})$, it suffices to take the restrictions of these functions to $[0,1]$. Things are not so trivial when one starts from smoother wavelet bases on the line. Assume that both ϕ and ψ have support width $2N-1$. In order to avoid having to deal with the two edges of $[0,1]$ at the same time, we can choose to start from the basis $\{\phi_{-j_0,k};\ k \in \mathbb{Z}\} \cup \{\psi_{j,k};\ j \le -j_0, k \in \mathbb{Z}\}$ for $L^2(\mathbb{R})$, where $2^{j_0-1} \ge N$ so that none of the functions has support straddling both 0 and 1. Even so there will be $2N-2$ wavelets, at every resolution level and at every end of $[0,1]$, that straddle an endpoint, so that their support is neither completely in $[0,1]$ nor completely in $\mathbb{R}\setminus]0,1[$. It is not a priori clear how to adapt them in such a way that the result is an orthonormal basis of $L^2([0,1])$.

Several solutions have been proposed for this problem. They all correspond to different choices of how to adapt the multiresolution hierarchy to the interval $[0,1]$.

1. Extending by zeros.

This solution consists in not doing anything at all. A function f supported on $[0,1]$ can always be extended to the whole line by putting $f(x)=0$ for $x\notin[0,1]$. This function can then be analyzed by means of the wavelets on the whole real line. There are two things wrong with this naive approach. First of all, this kind of extension typically introduces a discontinuity in f at $x=0$ or 1, which will be reflected by "large" wavelet coefficients for fine scales (i.e. wavelet coefficients which do not decay very fast) near the two edges, even if f itself is very smooth on $[0,1]$. The second "bad" aspect is that this approach uses "too" many wavelets. At scale $-j$, one finds $\langle f,\ \psi_{-j,k}\rangle\neq 0$ for typically 2^j+2N-1 wavelets; intuitively one should have to use only 2^j wavelets, at scale $-j$, when looking at problems on $[0,1]$.

2. Periodizing.

In this case, one expands a function f on $[0,1]$ into "periodized" wavelets defined by $\psi^{per}_{-j,k}(x)=2^{j/2}\ \sum_{\ell\in\mathbb{Z}}\ \psi(2^jx+2^j\ell-k)$, with $j\geq j_0\geq 0$ (for $j<0$, the $\psi^{per}_{-j,k}$ vanish identically), $0\leq k\leq 2^j-1$. These wavelets have to be supplemented by lowest resolution scaling functions $\phi^{per}_{-j_0,k}$, defined analogously; the result is an orthonormal basis of $L^2([0,1])$, associated with a multiresolution analysis in which V^{per}_{-j} is spanned by the $\phi^{per}_{-j,k}$. One now has exactly 2^j wavelets at scale $-j$, as well as 2^j scaling functions $\phi^{per}_{-j,k}$ in every V^{per}_{-j}. Since

$$\int_0^1 dx\ f(x)\ \psi^{per}_{-j,k}(x)=\int_{-\infty}^{\infty}dx\left[\sum_\ell f(x+\ell)\right]\ \psi_{-j,k}(x)\ ,$$

expanding a function on $[0,1]$ into periodized wavelets is equivalent to extending the original function into a periodic function with period 1 and analyzing this extension with the standard whole-line wavelets. Unless f was already periodic, this construction introduces again a discontinuity at $x=0$, $x=1$, which will show up as slow decay in the fine scale wavelet coefficients pertaining to the edges. Again, it will be impossible to characterize the one-sided regularity of f at 0 or 1 by looking at the decay of the $|\langle f,\ \psi^{per}_{j,k}\rangle|$ for $j\to-\infty$, unless f is periodic.

3. Reflecting at the edges.

In this case, one extends the function f on $[0,1]$ by mirroring it at 0 and 1; beyond -1 and 2 we mirror once more, and so on. The full extension is then defined by $f(x) = f(2n-x)$ if $2n-1 \le x \le 2n$, $f(x) = f(x-2n)$ if $2n \le x \le 2n+1$. If the original function on $[0,1]$ is continuous, then this extension will be continuous. Typically, however, the derivative of the extension has discontinuities at the integers. Expanding the "reflected" extension of a function on $[0,1]$ in a whole-line-basis of wavelets is equivalent to expanding the original function on $[0,1]$ with respect to "folded" wavelets $\psi_{j,k}^{\text{fold}}$ defined on $[0,1]$ by

$$\psi_{j,k}^{\text{fold}}(x) = \sum_{\ell\in\mathbb{Z}} \psi_{j,k}(x-2\ell) + \sum_{\ell\in\mathbb{Z}} \psi_{j,k}(2\ell - x) .$$

Starting from an orthonormal wavelet basis, this folding typically does not lead to an orthonormal wavelet basis on $[0,1]$. If ψ_{jk}, $\tilde{\psi}_{jk}$ are two biorthogonal wavelet bases, with ψ and $\tilde{\psi}$ both symmetric or antisymmetric around $1/2$, then their folded versions turn out to be still biorthogonal on $[0,1]$ however. The resulting biorthogonal multiresolution analysis hierarchies on $[0,1]$ have 2^j+1 (symmetric case) or 2^j (antisymmetric case) scaling functions and 2^j wavelets at resolution level j. Because the "reflected" extension typically has a discontinuous derivative, we can again not expect to characterize arbitrary regularity of f by means of the wavelet coefficients; decay of the $\langle f,\ \psi_{jk}^{\text{fold}}\rangle$ can characterize up to Lipschitz regularity (a gain over the two previous "solutions"), but not more, although one can do a little better by using two different pairs of biorthogonal bases. Explicitly, one finds, if the "original" (unfolded) ψ,ϕ are in C^r with $r > 1$, that a function f on $[0,1]$ is in $C^s([0,1])$, with $0 < s < 1$, if and only if

$$\sup_{\substack{j\ge 0 \\ 0\le k\le 2^j-1}} \quad 2^{j(s+1/2)} \left|\langle f, \tilde{\psi}_{-j,k}^{\text{fold}}\rangle\right| < \infty .$$

(For $s = 1$ a similar result holds, with C^1 replaced by a Zygmund-type space.) As usual, the "only if" part follows from $\int_0^1 dx\ \tilde{\psi}_{-j,k}^{\text{fold}}(x) = \int_{-\infty}^{\infty} dx\ \tilde{\psi}_{-j,k}(x) = 0$, the "if" part from the smoothness of ψ. If one tries to see what goes wrong if $s > 1$, say $1 < s < 2$, then the "only if" part would require $\int_0^1 dx\ x\tilde{\psi}_{-j,k}^{\text{fold}} = 0$, the "if" part $\psi \in C^r$ with $r > s$. The first requirement is equivalent with $\int_{-1}^1 dx\ (1-|x|)\tilde{\psi}(x-\ell) = 0$ for all $\ell \in \mathbb{Z}$, which is only possible if ϕ is the tent function $\phi(x) = 1-|x|$ if $|x| \le 1$, $\phi(x) = 0$ otherwise. But then $\psi \notin C^1$, and the "if" part fails. One can however characterize $f \in C^s$, $1 < s < 2$ if one uses two pairs of biorthogonal wavelet bases, one for the "if" part, one for the "only if" part. Values of $s > 2$ cannot be attained. For more details, see Cohen, Daubechies and Vial[1] (1992).

4. The construction of Y. Meyer.

A fourth solution was proposed in Meyer[8] (1992). The starting point of this construction is any one of the compactly supported bases in Daubechies[4] (1988), with N vanishing moments, and support ψ = support $\phi = [-N+1, N]$. The basis on $[0,1]$ constructed by Y. Meyer is derived from a multiresolution analysis that "lives" on $[0,1]$. At sufficiently fine scales, the approximation spaces $V_{-j}^{[0,1]}$ consist of $2^j - 2N + 2$ "interior" functions, $2N - 2$ "left edge" functions, and $2N - 2$ "right edge" functions. The complement spaces $W_{-j}^{[0,1]}$ are generated by $2^j - 2N - 2$ "interior" wavelets, $N - 1$ "left edge" wavelets, and $N - 1$ "right edge" wavelets. The total number of wavelets at scale j is thus 2^j, but the total number of scaling functions is larger, $2^j + 2N - 2$. The "interior" functions are simply those $\psi_{-j,k}$ or $\phi_{-j,k}$ (as they were defined on the whole line) which happen to have their support contained in $[0,1]$. The "edge" functions have to be constructed explicitly. In particular, the left edge functions $\phi_{-j,k}^{\text{left}}$ are obtained by orthonormalizing the $(2N-2)$ restrictions $\phi_{-j,k}|_{[0,1]}$ where k is chosen so that $0 \in$ interior support$(\phi_{-j,k})$. The right edge scaling functions are obtained similarly; the edge wavelets can then be computed from projections of those $\psi_{-j,k}|_{[0,1]}$ which straddle 0 or 1 and for which more than half the support is within $[0,1]$. (For details, see Meyer[8] (1992).) The result of the construction is an orthonormal family of wavelets in $[0,1]$, with N vanishing moments, and the same regularity as the original ψ; together with an orthonormal family of scaling functions on $[0,1]$ at the coarsest scale under consideration, these adapted wavelets constitute an orthonormal basis for $L^2([0,1])$. In addition, their regularity and vanishing moment properties ensure that they are inconditional wavelet bases for the Hölder spaces $C^s([0,1])$ for all $s < r$, where r is the regularity of the original wavelet basis, $\psi \in C^r$. Because the number of scaling functions at resolution j is larger than the number of wavelets, Meyer's construction cannot be generalized to wavelet packets on the interval: in a wavelet packet construction, wavelet coefficients get split as well as scaling coefficients, using the same filters, and for this it is essential that the two families have the same number of coefficients at every scale. That the number of scaling functions is not a power of 2 is also a nuisance for practical applications such as image analysis, where arrays are typically squares with 256×256 or 512×512 pixels. In order to implement the scheme, all the orthonormalization and projection matrices have to be computed explicitly. This involves the computation of integrals of the type

$$\int_0^\infty dx\ \phi(x+k)\phi(x+\ell) \quad \text{with} \quad -N+1 < k, \ell < N\ .$$

Using the refinement equation for ϕ, these can be computed by solving an $(N-1)(2N-3)$ dimensional linear system. This system is however very badly conditioned, because e.g. $\int_0^\infty dx\, |\phi(x-N+2)|^2 >>> \int_0^\infty dx\, |\phi(x+N-1)|^2$.

The disequilibrium among the $\int_0^\infty dx\, |\phi(x+k)|^2$ also expresses itself in other ways. One application of wavelet bases and multiresolution on the interval is the "natural" extension of functions living on the interval to functions on the whole line. Since the edge-wavelets and scaling functions can all be written as linear combinations of restrictions of whole-line functions, one can extend them trivially by "gluing on their tails back again", i.e. by replacing every $\phi_{-j,k}|_{[0,1]}$ by $\phi_{-j,k}$. If this is done for every edge term in the expansion of a function f on $[0,\infty)$, the result is a smooth function f^{ext} extending f to $\mathbb{R}$, with the appealing property that high frequency components in f spread out less to $(-\infty,0]$ than low frequency components. At any scale j, the extension is limited to $[-2^{-j}(2N-2),\infty)$. This doesn't work so well in practice, however: the extension of those edge scaling functions that are obtained from restricting $\phi_{-j,k}$ to $[0,1]$ which have only a tiny piece of their support in $[0,1]$ can have a huge amplitude outside $[0,1]$. This is the reason why B. Jawerth, in an application involving such extension operators for surface design in collaboration with B. Dahlberg, decided to develop a construction different from Meyer's. Another instance where one can feel the imbalance among the $\phi_{0,k}^{half}$ is in the plots of the edge functions. Typically, $\phi_{0,-N+1}|_{[0,\infty)}$ has much faster high amplitude oscillations than ϕ itself (the same oscillations are of course present in the tail of ϕ, but with exceedingly small amplitude); because of the orthonormalization procedure, this oscillatory behavior spreads to several edge scaling functions. Figure 1 shows the edge scaling functions for $N=4$, at the left side of the interval $[0,1]$; it illustrates this oscillatory behavior.

5. A different construction of interval wavelets.

This paper presents a fifth solution, also derived from compactly supported wavelet bases for $\mathbb{R}$. Like Meyer's construction, it uses "interior" and "edge" scaling functions at every resolution. We introduce fewer edge functions however, tailoring them so that the total number is exactly 2^j at resolution j; moreover, as in Meyer's case, all the polynomials on $[0,1]$ of degree $\leq N-1$ can be written as linear combinations of the scaling functions at any fixed scale. It then follows that all the corresponding wavelets, at the edge as well as in the interior, have N vanishing moments, and this is sufficient to ensure that we have again inconditional bases for the $C^s([0,1])$-spaces, with $s<r$ if $\psi \in C^r$. After completing our work, we learned that a similar construction was made independently by P. G. Lemarié-Rieusset[9] (1992), and by B. Jawerth.

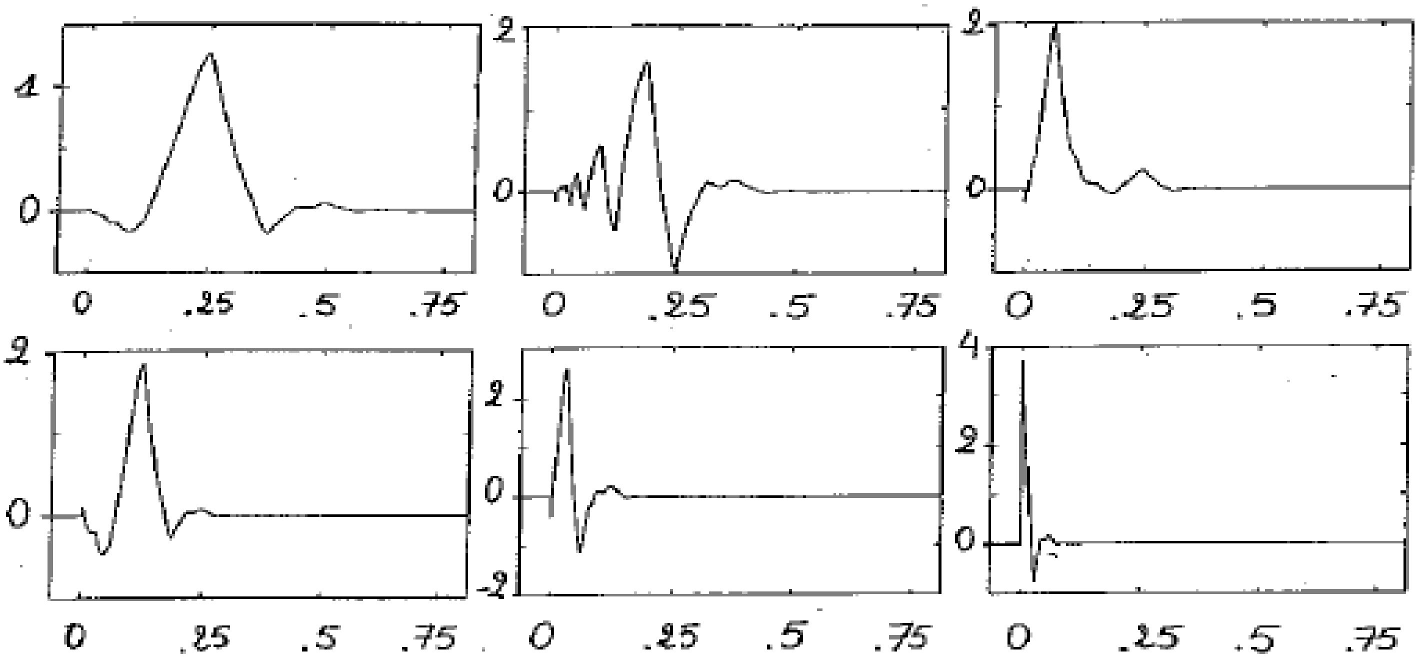

Figure 1: The adapted scaling functions in $V_{-3}^{[0,1]}$ at the left edge in the construction of Y. Meyer for $N = 4$.

A related construction, from the filter point of view, is in Herley, Kovačević, Ramchandran and Vetterli[10] (1992).

Our starting point is again the N vanishing moment family of Daubechies[4] (1988) or a variant (see Daubechies[5,6] (1990, 1992)). We choose to translate them so that support ϕ = support $\psi = [-N+1, N]$. Our goal is to retain the interior scaling functions, and to add adapted edge scaling functions in such a way that their union still generates all polynomials on $[0,1]$, up to a certain degree. Let us illustrate the principle of the construction by working on the half line $[0,\infty)$ instead of on $[0,1]$; we then only have to deal with the left edge, and it doesn't matter at which scale we work. The "interior" scaling functions at scale 0 are the $\phi_{0,k}$ with $k \geq N-1$; they are supported on $[0,\infty)$. By themselves, the interior $\phi_{0,k}$ do not even generate the constants on $[0,\infty)$, as is clear from $\phi_{0,k}(0) = \phi(-k) = 0$ for all $k \geq N-1$. Let us therefore add the constants "by hand". We define an edge function ϕ^0 by $\phi^0(x) = 1 - \sum_{k=N-1}^{\infty} \phi(x-k)$. The interior $\phi_{0,k}$ and this edge function ϕ^0 together generate all the constants on $[0,\infty)$. Moreover, because $\sum_{k=-\infty}^{\infty} \phi(x-k) = 1$, we also have, for $0 \leq x < \infty$, $\phi^0(x) = \sum_{k=-\infty}^{N-2} \phi(x-k) = \sum_{k=N+1}^{N-2} \phi(x-k)$, showing that ϕ^0 has compact support. It also shows, incidentally, that ϕ^0 is orthogonal to all the interior $\phi_{0,k}$. The only thing that we have to check is that by adding functions in this ad hoc way we don't leave the framework of a multiresolution hierarchy. We have however

$$\phi(x-k) = \sqrt{2} \sum_{\ell=2k-N+1}^{N+2k} h_{\ell-2k}\, \phi(2x-\ell)$$

and

$$\begin{aligned} \phi^0 &= \phi^0(2x) + \sum_{\ell=N-1}^{\infty} \phi(2x-\ell) \left[1 - \sqrt{2} \sum_{k=N-1}^{\infty} h_{\ell-2k}\right] \\ &= \phi^0(2x) + \sum_{\ell=N-1}^{3N-4} \phi(2x-\ell) \left[1 - \sqrt{2} \sum_{k=\lceil(\ell-N)/2\rceil}^{\lfloor(\ell+N-1)/2\rfloor} h_{\ell-2k}\right] , \end{aligned}$$

where we have used that $h_n = 0$ for $n < -N+1$ or $n > N$ and $\sum_n h_{2n} = \frac{1}{\sqrt{2}} = \sum_n h_{2n+1}$. It follows therefore that

$$V_0^{\text{left}} = \overline{\text{Span}\,\{\phi^0,\ \phi_{0,k};\ k \geq N-1\}} \subset \overline{\text{Span}\,\{\phi^0(2\cdot),\ \phi_{-1,k};\ k \geq N-1\}} = V_{-1}^{\text{left}} ;$$

similar inclusions hold immediately if we scale by other integer powers of 2, and we still have a hierarchy of nested spaces.

This is essentially all there is to the construction we propose here. If we want the edge + interior scaling functions to generate more polynomials than only the constants, then we have to add in,by hand, more edge functions (for the polynomials up to degree L, we add in total $L+1$ functions). If we work on the interval, then the same has to be done at the other edge as well. On the other hand, as pointed out earlier, for many applications it is desirable to have exactly 2^j scaling functions of scale j when working on $[0,1]$. Let us count how much room this leaves us for adding extra functions at the edges. If we start from a minimal support N-vanishing moment wavelet, then support $\phi = [-N+1, N]$, and for j sufficiently large we have exactly $2^j - 2N + 2$ interior scaling functions at scale j. This leaves room for adding $N-1$ ad hoc functions at each edge, so that the total family can generate polynomials of degree at most $N-2$. The unaltered whole-line scaling functions can generate all polynomials up to degree $N-1$, so that we seem to have "lost" one degree. In order to recover this one extra degree (and so be able to characterize the $C^s([0,1])$ spaces for the same range of s as we could on all of $\mathbb{R}$), we have to make room for one extra function at each edge of the interval. For this reason we abandon the two outermost interior scaling functions, which corresponds to retaining only the $\phi_{0,k}$ with $k \geq N$ rather than $k \geq N-1$ on the half line. More precisely, we define the N edge functions $\tilde{\phi}^k$, $k = 0, \ldots, N-1$, on $[0,\infty)$ by

$$\tilde{\phi}^k(x) = \sum_{n=0}^{2N-2} \binom{n}{k}\, \phi(x+n-N+1) . \tag{1}$$

These are all compactly supported, and their supports are staggered, i.e. support $\tilde{\phi}^k = [0, 2N-1-k]$; they are independent, and orthogonal to the $\phi_{0,m}$, $m \geq N$. Together with the $\phi_{0,m}$, $m \geq N$, they generate all the polynomials up to degree $N-1$ on $[0,\infty)$. Finally, there exist constants $a_{k,\ell}$, $b_{k,m}$ (which can be computed explicitly) so that

$$\tilde{\phi}^k(x) = \sum_{\ell=0}^{k} a_{k,\ell}\, \tilde{\phi}^\ell(2x) + \sum_{m=N}^{3N-2-2k} b_{k,m}\, \phi(2x-m)\ . \tag{2}$$

(For all proofs, see Cohen, Daubechies and Vial[1] (1992).)

One can obtain an orthonormal basis for V_0^{left} by orthonormalizing the $\tilde{\phi}^k$, since they are already orthogonal to the orthonormal $\phi_{0,m}$; scaling them leads to an orthonormal basis for every V_j^{left}. If one orthonormalizes by a Gramm-Schmidt procedure, starting with $\tilde{\phi}^{N-1}$, and working down to lower values of k, then the resulting orthonormal ϕ_k^{left}, $k = 0, \ldots, N-1$, still have staggered supports: support $\phi_k^{\mathrm{left}} = [0, N+k]$. To carry out the Gramm-Schmidt orthonormalization explicitly, we need again the overlap matrix $\langle \tilde{\phi}^k,\ \tilde{\phi}^\ell \rangle$. To compute this overlap matrix, we use the recurrence (2). For $k=0$, for instance, we have

$$\|\tilde{\phi}^0\|^2 = a_{0,0}^2\, \frac{1}{4} \|\tilde{\phi}^0\|^2 + \sum_{m=N}^{3N-2} b_{0,m}^2\, \frac{1}{4}\ ,$$

from which we obtain $\|\tilde{\phi}^0\|^2$. It then follows that

$$\langle \tilde{\phi}^0,\ \tilde{\phi}^1 \rangle = a_{0,0}\, a_{1,0}\, \frac{1}{4} \|\tilde{\phi}^0\|^2 + a_{0,0}\, a_{1,1}\, \frac{1}{4} \langle \tilde{\phi}^0,\ \tilde{\phi}^1 \rangle + \frac{1}{4} \sum_{m=N}^{3N-4} b_{0,m}\, b_{1,m}$$

leading to an explicit formula for $\langle \tilde{\phi}^0,\ \tilde{\phi}^1 \rangle$, since $\|\tilde{\phi}^0\|^2$ is known. One proceeds similarly for higher values of k.

The orthonormal ϕ_k^{left}, constructed with staggered supports along the lines indicated above, satisfy a recursion relation similar to (2) and inherited by all the scales j. Explicitly, there exist constants $H_{k,\ell}^{\mathrm{left}}$ and $h_{k,m}^{\mathrm{left}}$ (which can be computed explicitly from the $a_{k,\ell}$, $b_{k,\ell}$ in (2) and the orthonormalization procedure) such that

$$\phi_{-j,k}^{\mathrm{left}} = \sum_{\ell=0}^{N-1} H_{k,\ell}^{\mathrm{left}}\, \phi_{-j-1,\ell}^{\mathrm{left}} + \sum_{m=N}^{N+2k} h_{k,m}^{\mathrm{left}}\, \phi_{-j-1,m}\ . \tag{3}$$

All this was on the half line. If we work on the interval $[0,1]$, and we start with a scale fine enough so that the two edges don't interact, i.e. $2^j \geq 2N$, then there are $2^j - 2N$ interior scaling functions $\phi_{-j,N}, \ldots, \phi_{-j,2^j-N-1}$, and we add N functions at each end, following the principles outlined above. Together, these 2^j orthonormal functions span $V_{-j}^{[0,1]}$.

We now turn to the wavelets rather than the scaling functions. As usual, we define $W_{-j}^{[0,1]} = V_{-j-1}^{[0,1]} \cap \left(V_{-j}^{[0,1]}\right)^{\perp}$. From dimension counting, it immediately follows that $\dim W_{-j}^{[0,1]} = 2^j$. On the other hand it is easy to check that the $2^j - 2N$ functions $\psi_{-j,m}$, $m = N, \ldots, 2^j - N - 1$ are all in $W_{-j}^{[0,1]}$. Since they are all orthonormal, we therefore need to add an extra $2N$ wavelets (N at each edge) to provide an orthonormal basis for $W_{-j}^{[0,1]}$. How should they be constructed? To simplify notation, we return to the half line $[0, \infty)$. We define there $W_j^{\text{left}} = V_{j-1}^{\text{half}} \cap (V_j^{\text{half}})^{\perp}$; the $\psi_{j,m}$, $m \geq N$ all belong to W_j^{left}, and we are looking for N extra functions in W_j^{left}, orthonormal to these $\psi_{j,m}$. Define

$$\tilde{\psi}^k = \phi_{-1,k}^{\text{left}} - \sum_{m=0}^{N-1} \langle \phi_{-1,k}^{\text{left}},\ \phi_{0,m}^{\text{left}} \rangle \phi_{0,m}^{\text{left}} . \tag{4}$$

Then the $\tilde{\psi}^k$ are N independent functions in W_0^{left}, orthogonal to the $\psi_{0,m}$, $m \geq N$. Because of the recursion relation (3), the $\tilde{\psi}^k$ can be written as a linear combination of $\phi_{-1,\ell}^{\text{left}}$ and $\phi_{-1,m}$:

$$\tilde{\psi}^k = \sum_{\ell=0}^{k} c_{k,\ell}\ \phi_{-1,\ell}^{\text{left}} + \sum_{m=N}^{3N-2} d_{k,m}\ \phi_{-1,m} . \tag{5}$$

In a final step, these $\tilde{\psi}^k$ can be orthonormalized and we end up with an orthonormal family ψ_k^{left}, $k = 0, \ldots, N-1$. It is possible to orthonormalize in such a way that the ψ_k^{left} have staggered supports, support $\psi_k^{\text{left}} = [0, N+k]$. For any $j \in \mathbb{Z}$ we define again $\psi_{-j,k}^{\text{left}}(x) = 2^{j/2}\ \psi_k^{\text{left}}(2^j x)$; together with the $\psi_{-j,m}$, $m \geq N$, the $\psi_{-j,k}^{\text{left}}$, $k = 0, \ldots, N-1$ provide an orthonormal basis for W_{-j}^{left}; moreover, there exists constants $G_{k,\ell}^{\text{left}}$ and $g_{k,m}^{\text{left}}$ so that

$$\psi_{-j,k}^{\text{left}} = \sum_{\ell=0}^{N-1} G_{k,\ell}^{\text{left}}\ \phi_{-j-1,\ell}^{\text{left}} + \sum_{m=N}^{N+2k} g_{k,m}^{\text{left}}\ \phi_{-j-1,m} . \tag{6}$$

This completes our explicit construction, at least at a left end. The same has to be repeated at a right end. Combining the two leads to orthonormal bases for $W_{-j}^{[0,1]}$.

The result is an orthonormal basis for $L^2([0,1])$. If $\phi, \psi \in C^r$, this is also an unconditional basis for $C^s([0,1])$ for $s < r$. In particular, a bounded function f is in $C^s([0,1])$ if and only if

$$|\langle f,\ \psi_{-j,-k}^{\text{left}} \rangle|,\ |\langle f,\ \psi_{-j,m} \rangle|,\ |\langle f,\ \psi_{-j,2^j-N-k}^{\text{right}} \rangle| \leq C 2^{-j(s+1/2)} ,$$

where C is independent of j or m, k.

Figure 2 plots the scaling functions for $N = 4$, at the left end of $[0, \infty)$. Note that, like on the whole line, we have no explicit analytic expression for the wavelets and scaling functions on the interval; for practical applications, all that is really needed are the filter coefficients; in addition to the h_m, $g_m = (-1)^m h_{2N+1-m}$, we now also have the $H_{-k,-\ell}^{\text{left}}, h_{-k,m}^{\text{left}}, G_{-k,-\ell}^{\text{left}}, g_{-k,m}^{\text{left}}$ from (3) and

(6) (+ same at right). Tables for these filter coefficients can be found in Cohen, Daubechies and Vial[1] (1992). The adapted scaling functions in these plots are less oscillatory than those in §4. On [0, 1] the N functions ϕ_k^{left}, $k = 0, \ldots, N-1$, are pure polynomials (of degree $N-1$).

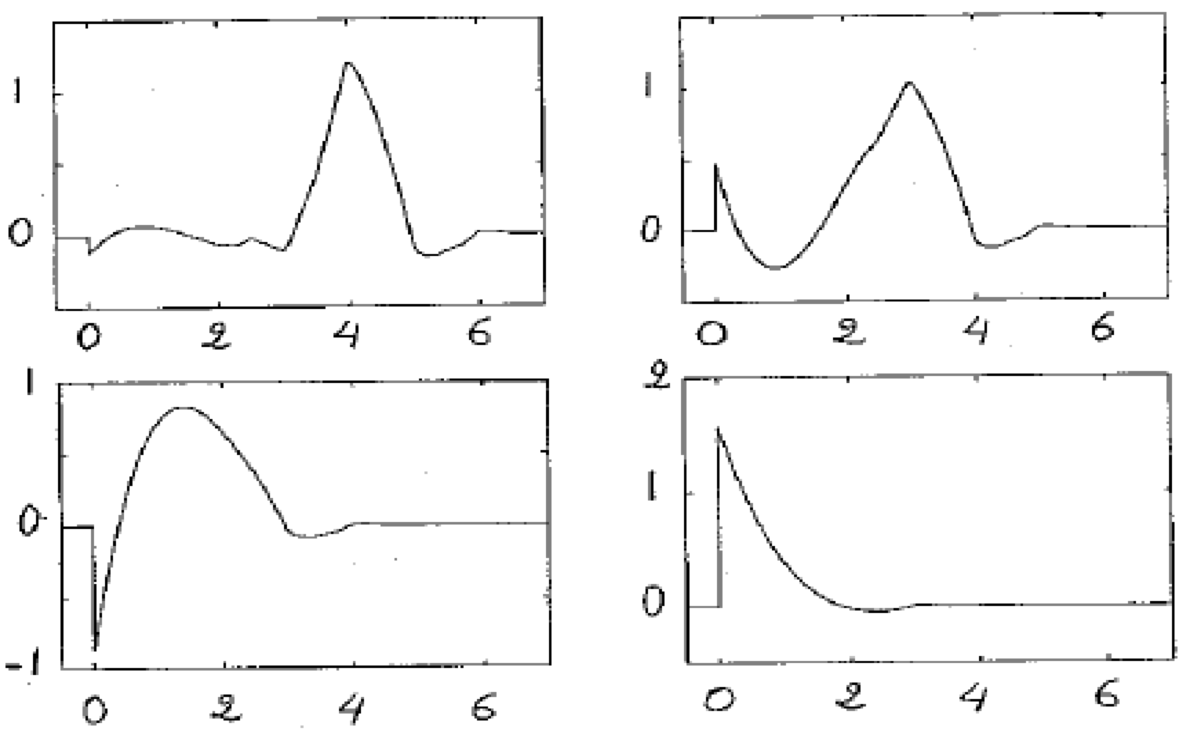

Figure 2: The adapted scaling functions in $V_0^{[0,\infty)}$ at the left edge in our new construction for $N = 4$.

This is because all the scaling functions together on $[0, \infty)$ generate the polynomials up to degree $N-1$; since the interior scaling functions $\phi_{0,m}$, $m \geq N$, only start kicking in from $x \geq 1$ onward, the N adapted scaling functions have to be polynomials themselves.

6. Discussion of the new construction.

Many variations are possible. One can, for instance, start from completely different families of whole-line wavelets, and adapt the number of additional edge scaling functions to their support width and number vanishing moments.

We have assumed that we want the scaling functions to generate all possible polynomials up to a certain degree. If the interval wavelets are used to solve a differential equation, then it may be useful to adapt the construction so that all the scaling functions and wavelets involved satisfy certain prescribed boundary conditions. P. Auscher[11] (1992) adapted the original construction by Y. Meyer in this way; his scheme carries over entirely to the present construction (with more numerical stability). The construction by P. G. Lemarié-Rieusset, which is essentially the same as ours, obtained independently, was carried out in view of this application.

The same ideas apply of course to biorthogonal wavelet bases. If one starts from a choice with (anti)symmetric wavelets and scaling functions, then the adapted scaling functions and wavelets at the right edge can be chosen to be the mirrors of their left edge equivalents. Since biorthogonality instead of orthonormality is wanted, there is more freedom in the choice of the edge functions, and one can optimize for extra criteria.

There is an important difference between wavelets on the line and wavelets on $[0,1]$, which results in the necessity, in at least some applications, to precondition the data (e.g. an image) prior to their wavelet decomposition. Scaling functions on all of $\mathbb{R}$ have the property $\int dx\ \phi_{-j,k}(x) = 2^{-j/2}$, independently of k. A consequence of this is that the corresponding low pass filter preserves the sequence $\ldots 1111 \ldots$. For the specially adapted scaling functions at the edge of $[0,1]$, we typically have $\int_0^1 dx\ \phi^{\text{edge}}_{-j,k}(x) \neq 2^{-j/2}$. The result is that the sequence invariant under low pass filtering is not $111 \ldots 111$, but rather a sequence consisting of only 1-s in the middle, but with different initial and final entries. Something similar happens for sequences corresponding to higher degree polynomials. In practical examples (e.g. images) one still would like simple polynomial sequences like $1\ 1\ 1\ 1 \ldots$ or $1\ 2\ 3\ 4 \ldots$ to lead to a zero high-pass component, however. This can still be achieved if we perform a prefiltering on the data. The details of this scheme can be found in Cohen, Daubechies and Vial[1] (1992).

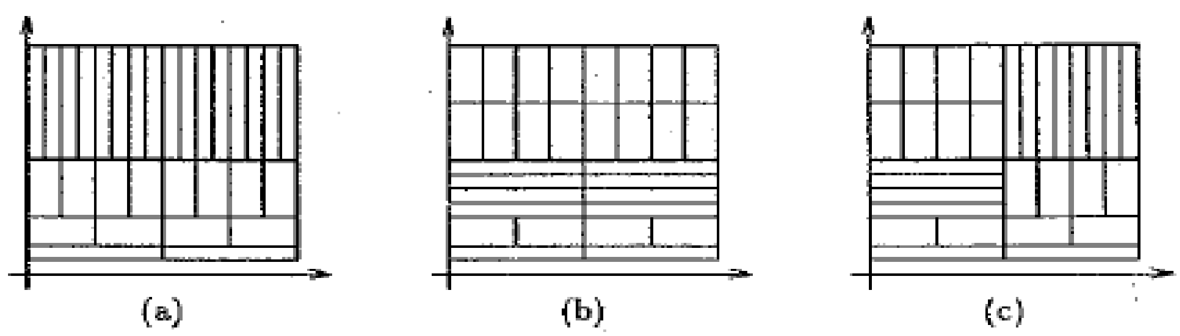

Figure 3: Different time frequency tilings.

Apart from the obvious applications mentioned above (image analysis, solving p.d.e-s with boundary conditions on an interval or a box), constructions of wavelets on the interval can also be used for e.g. nonuniform tilings of the time frequency plane. This is illustrated in Fig. 3: Fig. 3a shows the standard wavelet tiling; Fig. 3b the tiling resulting from another wavelet packet basis; the tiling in Fig. 3c is obtained by choosing different wavelet packet bases on consecutive intervals. A straightforward application of wavelet packets derived from the construction in §5 leads to abrupt cut-offs between the intervals; in order to obtain smoother transitions one has to develop taper-off techniques that let the different intervals "inter-penetrate" to some extent (preferably the penetration should be proportional to the support of the different wavelet packets). This last is work in progress. Another possible application of wavelets on the interval lies in the extension operators mentioned above (where every $\phi_{-j,k}|_{[0,1]}$ gets extended to its natural extension $\phi_{-j,k}$); such extensions could be used e.g. to avoid boundary problems in continuous wavelet transforms of data confined to a finite interval.

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