SIAM J. MATH. ANAL.
Vol. 22, No. 2, pp. 554-572, March 1991

016

A SIMPLE WILSON ORTHONORMAL BASIS WITH EXPONENTIAL DECAY*

INGRID DAUBECHIES†, STÉPHANE JAFFARD‡, AND JEAN-LIN JOURNÉ§

Abstract. Following a basic idea of Wilson ["Generalized Wannier functions," preprint] orthonormal bases for $L^2(\mathbb{R})$ which are a variation on the Gabor scheme are constructed. More precisely, $\phi \in L^2(\mathbb{R})$ is constructed such that the ψ_{ln}, $l \in \mathbb{N}$, $n \in \mathbb{Z}$, defined by

$$\begin{aligned}\psi_{0n}(x) &= \phi(x-n)\\ \psi_{ln}(x) &= \sqrt{2}\,\phi\left(x-\frac{n}{2}\right)\cos(2\pi l x) \quad \text{if } l \neq 0,\ l+n \in 2\mathbb{Z}\\ &= \sqrt{2}\,\phi\left(x-\frac{n}{2}\right)\sin(2\pi l x) \quad \text{if } l \neq 0,\ l+n \in 2\mathbb{Z}+1,\end{aligned}$$

constitute an orthonormal basis. Explicit examples are given in which both ϕ and its Fourier transform $\hat{\phi}$ have exponential decay. In the examples ϕ is constructed as an infinite superposition of modulated Gaussians, with coefficients that decrease exponentially fast. It is believed that such orthonormal bases could be useful in many contexts where lattices of modulated Gaussian functions are now used.

Key words. orthonormal bases, phase space localization, time-frequency analysis

AMS(MOS) subject classifications. 46C10, 81C40, 94A11

1. Introduction. In several applications in quantum mechanics and in signal analysis, sets of functions generated from one single function by phase space translations are encountered:

$$g_{mn}(x) = e^{2\pi i \alpha m x} g(x-\beta n), \qquad m, n \in \mathbb{Z}. \tag{1.1}$$

If the function g and its Fourier transform $\hat{g}$,

$$\hat{g}(\xi) = \int dx\, e^{2\pi i x \xi} g(x),$$

are both centered around zero, then the function g_{mn} is centered around the phase space point $(\alpha m, \beta n)$. We can then hope to use the functions g_{mn} for expansions of functions with good phase space localization. More concretely, we would like expansions of the type

$$f = \sum_{m,n} c_{mn}(f) g_{mn}, \tag{1.2}$$

with the property that the $c_{mn}(f)$ are nonnegligible only for those values of (m, n) associated to phase space points where f is nonnegligible. For example, if $\int_{|t|\geqq T} dt\, |f(t)|^2 \leqq \varepsilon \|f\|^2$ and $\int_{|\xi|\geqq\Omega} d\xi\, |\hat{f}(\xi)|^2 = \varepsilon \|f\|^2$, then we would prefer most of the "content" of f to be concentrated in the $c_{mn}(f)$ with $(m\alpha, n\beta)$ within or close to the rectangle $[-\Omega, \Omega] \times [-T, T]$. More concretely, this can be translated into the

* Received by the editors September 5, 1989; accepted for publication (in revised form) March 13, 1990.

† AT&T Bell Laboratories, 600 Mountain Avenue, Murray Hill, New Jersey 07974. This author is "Bevoegdverklaand Navonsen" at the Belgian National Science Foundation (on leave); also on leave from the Department of Theoretical Physics, Vrije Universiteit, Brussels, Belgium.

‡ Centre d'Etude et de Recherche en Mathématique Appliquée, Ecole des Ponts et Chaussées, La Courtine, 93167 Noisy-le-Grand, France.

§ Princeton University, Princeton, New Jersey 08544. This author's work was supported by the National Science Foundation.

requirement

$$\sum_{\substack{|\alpha m| \geqq T+\Delta T \\ |\beta n| \geqq \Omega+\Delta\Omega}} |c_{mn}(f)|^2 \leqq C\varepsilon \|f\|^2, \tag{1.3}$$

where C should be independent of ε, T, and Ω, and where ΔT, $\Delta\Omega$ should only depend on the desired precision ε.

One example of a set of functions of type (1.1) are the phase space Wannier functions used in solid state physics. In the absence of a potential they are obtained by the choice

$$g(x) = \frac{\sin \pi x}{\pi x},$$

which corresponds to

$$\hat{g}(\xi) = \begin{cases} 1, & |\xi| < \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

(When the potential is nonzero, the Wannier functions are more complicated [1].) For this choice of g, and for the parameter choice $\alpha = \beta = 1$, the functions (1.1) constitute an orthonormal basis of $L^2(\mathbb{R})$. Expansions of type (1.2) are therefore simple to obtain: it suffices to take $c_{mn}(f) = \int dx\, \overline{g_{mn}(x)}\, f(x)$. Unfortunately, the localization of g is not very good. The function g has a rather long tail, so that

$$\int dx\, x^2 |g(x)|^2 = \infty.$$

As a consequence of this, expansions of functions with respect to the phase space Wannier functions do not have the good phase space localization features described above.

Another example of a set of functions of type (1.1) is given by the "Gabor expansions." These correspond to the choice

$$g(x) = 2^{1/4} \exp(-\pi x^2).$$

In the original proposal of Gabor [2], the parameter choice $\alpha = \beta = 1$ is made. Unfortunately, this parameter choice leads to numerically unstable expansions: for any $\varepsilon > 0$, there exists $f \in L^2(\mathbb{R})$ such that $\|f\| = 1$ but $\sum_{m,n} |c_{mn}(f)|^2 \leqq \varepsilon$. It can be shown that this phenomenon happens for any choice of α, β such that $\alpha\beta = 1$ [3a, b], [4a, b]. If $\alpha\beta > 1$, then the g_{mn} do not span all of $L^2(\mathbb{R})$ [5], [6]. If $\alpha\beta < 0.996$, then numerically stable expansions of type (1.2) do exist, with the "good" phase space localization described by (1.3) (see [7], [8]; it is conjectured that this situation persists for $\alpha\beta < 1$). There is, however, a price to pay: for $\alpha\beta < 1$, the g_{mn} are highly redundant, in the sense that any finite number of them lies in the closed linear span of all the others. While Gabor expansions with $\alpha\beta < 1$ are indeed used in practical computations in atomic and nuclear physics, this redundancy can be quite a nuisance.

These two examples illustrate how convenient it would be to have a nice orthonormal basis ($\rightarrow$ no redundancy) of type (1.1), based on a function g such that both g and $\hat{g}$ have good decay properties ($\rightarrow$ expansions with good phase space localization). Unfortunately, such an orthonormal basis does not exist. A theorem stated by Balian [9] and Low [10] asserts that a set of functions of type (1.1) can only constitute an orthonormal basis if either $\int dx\, x^2 |g(x)|^2 = \infty$ or $\int d\xi\, \xi^2 |\hat{g}(\xi)|^2 = \infty$. Balian's and Low's proofs contain a technical gap that was filled by Coifman and Semmes, as reported in [7]; a much simpler proof was subsequently found by Battle [11]. Even if the orthonormality, but not the "basis" requirement, is given up, the same conclusion still holds, as shown by the extension of Battle's argument in [12]. Both the original proof and Battle's proof of the Balian–Low theorem rely heavily on the special structure of the g_{mn} as defined by (1.1). We might therefore wonder whether there exist more general bases, $\psi_{mn}(x)$, with phase space localizations distributed more or less regularly over phase space, and such that uniform bounds on the decay of all the ψ_{mn} and $(\psi_{mn})^\wedge$, away from their central value, would hold. It turns out that there is indeed improvement from giving up the simplicity of (1.1), but only very little. Bourgain [13] has constructed an orthonormal basis of ψ_{mn} such that

$$\begin{aligned}&\int dx\,(x-\bar{x}_{mn})^2|\psi_{mn}(x)|^2 \leqq C,\\ &\int d\xi\,(\xi-\bar{\xi}_{mn})^2|(\psi_{mn})^\wedge(\xi)|^2 \leqq C,\end{aligned} \tag{1.4}$$

uniformly in m, n, where $\bar{x}_{mn}=\int dx\,x|\psi_{mn}(x)|^2$, and $\bar{\xi}_{mn}$ is defined analogously. However, as soon as a slightly sharper localization is required, we hit another no-go-theorem, even for these more general constructions: Steger [14] proved that $L^2(\mathbb{R})$ does not admit an orthonormal basis ψ_{mn} satisfying

$$\begin{aligned}&\int dx\,(x-\bar{x}_{mn})^{2(1+\varepsilon)}|\psi_{mn}(x)|^2 \leqq C,\\ &\int d\xi\,(\xi-\bar{\xi}_{mn})^{2(1+\varepsilon)}|(\psi_{mn})^\wedge(\xi)|^2 \leqq C.\end{aligned} \tag{1.5}$$

Orthonormality, or, what is weaker, the existence of numerically stable expansions of type (1.2) with nonredundant functions ψ_{mn}, is therefore incompatible with good phase space localization.

In all the above, "good phase space localization" stands for strong decay properties of the ψ_{mn}, $(\psi_{mn})^\wedge$ away from the average values $\bar{x}_{mn}$, $\bar{\xi}_{mn}$. This corresponds to a picture in which both ψ_{mn} and $(\psi_{mn})^\wedge$ have essentially *one* peak. In [15] Wilson proposes instead to construct orthonormal bases ψ_{mn} of the type

$$\psi_{mn}(x)=f_m(x-n), \qquad m\in\mathbb{N}, \quad n\in\mathbb{Z}, \tag{1.6}$$

where $\hat{f}_m$ has *two* peaks, situated near $m/2$ and $-m/2$,

$$\hat{f}_m(\xi)=\phi_m^+\left(\xi-\frac{m}{2}\right)+\phi_m^-\left(\xi+\frac{m}{2}\right), \tag{1.7}$$

with ϕ_m^+, ϕ_m^- centered around zero. He proposes numerical evidence for the existence of such an orthonormal basis, with uniform exponential decay for f_m and ϕ_m^+, ϕ_m^-. In his numerical construction he further "optimizes" the localization by requiring

$$\int d\xi\,\xi^2\overline{(\psi_{mn})^\wedge(\xi)}\,\psi_{m'n'}(\xi)=0 \quad \begin{array}{l}\text{if } |m-m'|>1,\\ \text{or if } |m-m'|=1,\ |n-n'|>1.\end{array} \tag{1.8}$$

In [16] Sullivan et al. present arguments explaining both the existence of Wilson's basis and its exponential decay. In both [15] and [16] there are infinitely many functions $\phi_m^\pm$; as m tends to ∞, the $\phi_m^\pm$ tend to a limit function $\phi_\infty^\pm$.

The moral of Wilson's construction is that orthonormal bases with good phase space localization are possible after all if bimodal functions as in (1.7) are used. This is reminiscent of what happens for orthonormal wavelet bases, i.e., orthonormal bases of $L^2(\mathbb{R})$ of the type

$$h_{mn}(x)=2^{-m/2}h(2^{-m}x-n), \qquad m, n\in\mathbb{Z}. \tag{1.9}$$

There exist functions h with excellent phase space localization properties such that the functions (1.9) constitute an orthonormal basis. In [17] Meyer constructs such a function h with compactly supported, C^∞ Fourier transform $\hat{h}$; [18]-[20] give examples of exponentially decaying $h \in C^k$; and [21] constructs compactly supported $h \in C^k$. In all these examples, $|\hat{h}|$ has two peaks, one for positive and one for negative frequencies. It has been shown [22] that these two peaks need not be symmetrical in order for the h_{mn} to constitute an orthonormal basis (the examples in [17]-[21] all have symmetric peaks for $|\hat{h}|$). However, there is no example, so far, of reasonably well-localized functions $h^\pm$ such that support $(h^\pm) \subset \mathbb{R}_\pm$ and such that the $h^\pm_{mn}$ constitute an orthonormal basis of $L^2(\mathbb{R})$, corresponding to wavelet bases with only one "peak" in frequency. (Equivalently, there is no example of a reasonably smooth function $\phi = \hat{h}^+$ such that the functions $2^{m/2} \exp(2\pi i 2^m n\xi)\phi(2^m\xi)$ are an orthonormal basis of $L^2(\mathbb{R}_+)$.) It is believed, without proof so far, that no such basis exists. This seems to be the analogue, for the wavelet situation, of the Balian–Low theorem.

In this paper we construct an explicit bimodal orthonormal basis of the type (1.6), (1.7). Our basis is especially simple because it is again generated by one single function, unlike the bases in [15], [16]. More explicitly, we construct a real function ϕ such that with the definitions

$$\hat{f}_1(\xi) = \phi(\xi),$$

$$\text{(1.9a)} \qquad \hat{f}_{2\ell+\kappa}(\xi) = \frac{1}{\sqrt{2}}[\phi(\xi-\ell) + (-1)^{\ell+\kappa}\phi(\xi+\ell)]\, e^{i\pi\kappa\xi}, \qquad \ell \in \mathbb{N}\backslash\{0\}, \quad \kappa = 0 \text{ or } 1,$$

the family

$$\text{(1.9b)} \qquad \psi_{mn}(x) = f_m(x-n), \qquad m \in \mathbb{N}\backslash\{0\}, \quad n \in \mathbb{Z}$$

constitutes an orthonormal basis. Both ϕ and its Fourier transform $\hat{\phi}$ have exponential decay. Moreover, ϕ can be explicitly constructed as a rapidly converging superposition of Gaussians. All these features should make the basis constructed here especially attractive for the computations in atomic and nuclear physics where the Gabor functions are now used. The price we pay for the simplicity of our Wilson basis is that the near-diagonalization (1.8) of ξ^2 no longer holds.

This paper is organized as follows. In § 2 we derive necessary and sufficient conditions on ϕ for the ψ_{mn}, defined by (1.9), to be an orthonormal basis of $L^2(\mathbb{R})$. In § 3 we rewrite these conditions in another form, via the Zak transform. In their new form, it is easy to see how to satisfy these conditions. We use this in § 4 to construct an explicit Wilson basis with all the properties mentioned above. It turns out that our construction is related to "tight frames" [23], [7]. We review this concept in § 5, and explain how it is linked to the present construction. This leads to an alternate construction method, given in § 6, which is easier to implement numerically. Finally § 7 gives some concluding remarks. In particular, we show how a relabelling of the ψ_{mn} in (1.9) reduces the construction to the formula given in the Abstract.

2. Necessary and sufficient conditions. It suffices to prove that

$$\text{(2.1)} \qquad \|\psi_{mn}\| = 1, \qquad m \in \mathbb{N}\backslash\{0\}, \quad n \in \mathbb{Z},$$

and

$$\text{(2.2)} \qquad \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \langle g, \psi_{mn}\rangle\langle\psi_{mn}, h\rangle = \langle g, h\rangle$$

for all $g, h \in L^2(\mathbb{R})$. Indeed, from (2.1) and (2.2) we obtain

$$\begin{aligned} 1 = \|\psi_{m'n'}\|^2 &= \sum_{m,n} |\langle \psi_{m'n'}, \psi_{mn}\rangle|^2 \\ &= 1 + \sum_{(m,n)\neq(m',n')} |\langle \psi_{m'n'}, \psi_{mn}\rangle|^2, \end{aligned}$$

whence $\langle \psi_{mn}, \psi_{m'n'}\rangle = \delta_{mm'}\delta_{nn'}$. It follows that (2.1) and (2.2) imply that ψ_{mn} constitute a total orthonormal set.

We first concentrate on (2.2). Using Parseval's identity and the Poisson summation formula, we find

$$\sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \langle g, \psi_{mn}\rangle\langle \psi_{mn}, h\rangle = \sum_{m=1}^{\infty} \sum_{k=-\infty}^{\infty} \int d\xi\, \overline{\hat{g}(\xi)}\, \hat{h}(\xi+k)\hat{f}_m(\xi)\overline{\hat{f}_m(\xi+k)}.$$

(Note that we use the physicist's convention for the inner product in $L^2(\mathbb{R})$, which is linear in the *second* argument, $\langle g, h\rangle = \int dx\, \overline{g(x)}h(x)$.) In order to have (2.2), it is therefore necessary and sufficient that

$$\sum_{m=1}^{\infty} \hat{f}_m(\xi)\overline{\hat{f}_m(\xi+k)} = \delta_{k0}. \tag{2.3}$$

Let us write this out in terms of ϕ. For the time being, we disregard any convergence questions; for the function we will construct all series converge absolutely and uniformly. We also assume ϕ to be real.

$$\begin{aligned} &\sum_{m=1}^{\infty} \hat{f}_m(\xi)\overline{\hat{f}_m(\xi+k)} \\ &\quad = \phi(\xi)\phi(\xi+k) + \frac{1}{2}\sum_{\ell=1}^{\infty}\sum_{\kappa=0}^{1} [\phi(\xi-\ell) + (-1)^{\ell+\kappa}\phi(\xi+\ell)] \\ &\quad\quad \cdot [\phi(\xi - \ell + k) + (-1)^{\ell+\kappa} \phi(\xi+\ell+k)] e^{i\pi\kappa k} \\ &\quad = \phi(\xi)\phi(\xi+k) + \sum_{\ell\in\mathbb{Z},\ell\neq 0} \phi(\xi+\ell)\phi(\xi+\ell+k)\frac{1}{2}(1+(-1)^k) \\ &\quad\quad + \sum_{\ell\in\mathbb{Z},\ell\neq 0} (-1)^\ell\phi(\xi+\ell)\phi(\xi-\ell+k)\frac{1}{2}(1-(-1)^k). \end{aligned} \tag{2.4}$$

If k is even, $k = 2j$, then

$$(2.4) = \sum_{\ell\in\mathbb{Z}} \phi(\xi+\ell)\phi(\xi+\ell+2j).$$

If k is odd, $k = 2j+1$, then

$$(2.4) = \sum_{\ell\in\mathbb{Z}} (-1)^\ell\phi(\xi+\ell)\phi(\xi-\ell+2j+1) = 0,$$

as is easily shown by the change of summation index $\ell' = -\ell + 2j + 1$. It follows that (2.3) is equivalent to

$$\sum_{\ell\in\mathbb{Z}} \phi(\xi+\ell)\phi(\xi+\ell+2j) = \delta_{j0}. \tag{2.5}$$

We now turn to (2.1). It clearly suffices to prove $\|\hat{f}_m\| = 1$, $m \in \mathbb{N}\backslash\{0\}$. For $m = 1$ this gives

$$\int d\xi\, |\phi(\xi)|^2 = 1.$$

For $m=2\ell+\sigma$, $\ell \geqq 1$, we find

$$\|\hat{f}_m\|^2=\frac{1}{2}\int d\xi\,|\phi(\xi-\ell)+(-1)^{\ell+\sigma}\phi(\xi+\ell)|^2$$

$$=1+(-1)^{\ell+\sigma}\int d\xi\,\phi(\xi-\ell)\phi(\xi+\ell).$$

It follows that (2.1) is equivalent to

$$\int d\xi\,\phi(\xi)\phi(\xi+2\ell)=\delta_{\ell 0}. \tag{2.6}$$

This condition is automatically satisfied if (2.5) holds:

$$\int_{-\infty}^{\infty} d\xi\,\phi(\xi)\phi(\xi+2\ell)=\sum_{k\in\mathbb{Z}}\int_0^1 d\xi\,\phi(\xi+k)\phi(\xi+k+2\ell)$$

$$=\int_0^1 d\xi\,\delta_{\ell 0}=\delta_{\ell 0}.$$

We have again assumed that ϕ is sufficiently well behaved so that the summation and integration may be commuted in this computation. It is easily checked that it is sufficient that ϕ decays faster than $|\xi|^{-1}$ for $|\xi|\to\infty$. For the examples we will construct, this is no problem. The following proposition summarizes our findings.

PROPOSITION 2.1. *Suppose that ϕ is a real function on $\mathbb{R}$ satisfying*

$$|\phi(\xi)|\leqq C(1+|\xi|)^{-1-\varepsilon}$$

for some C, $\varepsilon>0$. Then the functions ψ_{mn} defined by (1.9) *constitute an orthonormal basis for $L^2(\mathbb{R})$ if and only if*

$$\sum_{\ell\in\mathbb{Z}}\phi(\xi+\ell)\phi(\xi+\ell+2j)=\delta_{j0}.$$

We therefore have only one set of conditions, namely (2.5). This condition can be almost trivially satisfied if we choose ϕ to be supported in $[-1,1]$. In this case $\phi(\xi)\phi(\xi+2\ell)=0$ if $\ell\neq 0$, for any $\xi\in\mathbb{R}$. It follows that (2.5) is satisfied if $\sum_\ell \phi(\xi+\ell)^2=1$. Since this sum is periodic in ξ with period 1, we only need to check what happens for $0\leqq\xi\leqq 1$. For ϕ supported in $[-1,1]$, this means we only need to ascertain that $\phi(\xi)^2+\phi(\xi-1)^2=1$ for $0\leqq\xi\leqq 1$. Such ϕ are easy to construct: for any function F such that

$$F:\mathbb{R}\to\mathbb{R}$$

$$F(x)=\begin{cases}0, & x\leqq 0,\\ 1, & x\geqq 1,\end{cases}$$

$$0\leqq F(x)\leqq 1 \quad \text{for all } x,$$

the function ϕ defined by

$$\phi(\xi)=\begin{cases}\sin\left[\frac{\pi}{2}F(\xi+1)\right], & \xi\leqq 0,\\ \cos\left[\frac{\pi}{2}F(\xi)\right], & \xi\geqq 0\end{cases}$$

is a function supported in $[-1, 1]$ which satisfies (2.5), hence (2.1) and (2.2). If F is C^k (where k may be ∞), then ϕ is C^k. The corresponding ψ_{mn} are C^∞-functions; their decay at ∞ is regulated by the regularity of F. If F is C^∞, then the ψ_{mn} have "fast decay," i.e., for all $N \in \mathbb{N}$, there exists C_N such that

$$|\psi_{mn}(x)| \leqq C_N(1+|x-n|^2)^{-N}.$$

In practice, however, the constants C_N turn out to be rather large, so that the *numerical* localization of the ψ_{mn} is not very good. The examples we construct in § 4, corresponding to noncompactly supported ϕ, have better effective localization.

3. The Zak transform—rewriting the conditions. Using a unitary transformation, we will rewrite the infinitely many conditions (2.5) (one for every j) into a different form, reducing them to one single condition which is then easy to satisfy. The unitary map we shall use is the Zak transform. For the purposes of this paper, we define the Zak transform by

$$(U_Z g)(t, s) = \sqrt{2} \sum_{k \in \mathbb{Z}} e^{2\pi i t k} g(2(s-k)). \tag{3.1}$$

This is well defined for functions g with sufficient decay, $|g(x)| \leqq C(1+|x|^2)^{-1/2-\varepsilon}$. The two-variable function $G = U_Z g$ is periodic in the first and "semi-periodic" in the second variable,

$$\begin{aligned} G(t+1, s) &= G(t, s), \\ G(t, s+1) &= e^{2\pi i t} G(t, s). \end{aligned} \tag{3.2}$$

The set of all functions G of two variables satisfying the periodicity conditions (3.2) can be equipped with the norm

$$\|G\|^2 = \int_0^1 dt \int_0^1 ds\, |G(t, s)|^2. \tag{3.3}$$

We will denote the closure of this set, under the norm (3.3), by $\mathscr{Z}$. A function G is in $\mathscr{Z}$ if and only if its restriction to $[0, 1[\times [0, 1[$ is square integrable, and it satisfies the periodicity conditions (3.2) almost everywhere. It follows that $\mathscr{Z}$ is isomorphic with $L^2([0, 1]^2)$. The functions $E_{mn}(t, s)$, defined by

$$E_{mn}(t, s) = e^{2\pi i n t}\, e^{2\pi i m s} \quad \text{for } t, s \in [0, 1[,$$

extended by (3.2) to all of $\mathbb{R}^2$, constitute an orthonormal basis for $\mathscr{Z}$.

The map U_Z defined by (3.1) can be extended to a unitary map from $L^2(\mathbb{R})$ to $\mathscr{Z}$. This follows from the fact that U_Z maps the orthonormal basis $e_{mn}(x) = e^{\pi i m x}\chi(x-2n)$, where $\chi(x) = 2^{-1/2}$ if $0 \leqq x < 2$, $\chi(x) = 0$ otherwise, to the orthonormal basis E_{mn} of $\mathscr{Z}$, $U_Z e_{mn} = E_{mn}$.

The Zak transform has many interesting properties; it derives its name from its systematic study by J. Zak, who introduced it as a tool in solid state physics [24a-c]. It had already been studied sporadically before Zak's work, and it is claimed that even Gauss was already aware of some of its properties. An excellent review of the mathematical properties of U_Z and its applications to signal analysis is Janssen's paper [25], which also contains an extensive reference list.

The inverse transform of (3.1) is given by

$$(U_Z^{-1} G)(x) = \frac{1}{\sqrt{2}} \int_0^1 dt\, G\left(t, \frac{x}{2}\right). \tag{3.4}$$

Again this is only well defined for some G in $\mathscr{Z}$ (including all bounded G), but it can be extended to all of $\mathscr{Z}$.

There exists a relationship between $U_Z g$ and $U_Z \hat{g}$. Using the Poisson summation formula, we find

$$(U_Z\hat{g})(t,s)=\sqrt{2}\sum_{\ell\in\mathbb{Z}} e^{2\pi it\ell}\int_{-\infty}^{\infty} dx\, e^{4\pi i(s-\ell)x} g(x)$$

$$=\frac{1}{\sqrt{2}}e^{2\pi ist}\sum_{k\in\mathbb{Z}} e^{-2\pi isk} g\left(\frac{t-k}{2}\right) \tag{3.5a}$$

$$=\frac{1}{2}\sum_{j=0}^{3} e^{2\pi is(t+j)}(U_Z g)\left(-4s,\frac{t+j}{4}\right).$$

Similarly,

$$(U_Z g)(t,s)=\frac{1}{2}\sum_{j=0}^{3} e^{2\pi is(t+j)}(U_Z\hat{g})\left(4s,-\frac{t+j}{4}\right). \tag{3.5b}$$

Let us now apply all this to the problem at hand. We define $\Phi=U_Z\phi$, and we rewrite (2.5) in terms of Φ. We have

$$\sum_{\ell\in\mathbb{Z}}\phi(\xi+\ell)\phi(\xi+\ell+2j)=\frac{1}{2}\sum_{\ell\in\mathbb{Z}}\int_0^1 dt\int_0^1 dt'\,\Phi\left(t,\frac{\xi+\ell}{2}\right)\Phi\left(t',\frac{\xi+\ell}{2}+j\right)$$

$$=\frac{1}{2}\sum_{k\in\mathbb{Z}}\int_0^1 dt\int_0^1 dt'\left[\Phi\left(t,\frac{\xi}{2}+k\right)\Phi\left(t',\frac{\xi}{2}+k+j\right)\right.$$

$$\left.+\Phi\left(t,\frac{\xi+1}{2}+k\right)\Phi\left(t',\frac{\xi+1}{2}+k+j\right)\right]$$

$$=\frac{1}{2}\sum_{k\in\mathbb{Z}}\int_0^1 dt\int_0^1 dt'\, e^{2\pi ik(t+t')}\, d^{2\pi ijt'}$$

$$\cdot\left[\Phi\left(t,\frac{\xi}{2}\right)\Phi\left(t',\frac{\xi}{2}\right)+\Phi\left(t,\frac{\xi+1}{2}\right)\Phi\left(t',\frac{\xi+1}{2}\right)\right]$$

(use (3.2))

$$=\frac{1}{2}\int_0^1 dt\, e^{-2\pi ijt}\left[\left|\Phi\left(t,\frac{\xi}{2}\right)\right|^2+\left|\Phi\left(t,\frac{\xi+1}{2}\right)\right|^2\right].$$

In the last step we have assumed that $\Phi(\cdot,s)$ is square integrable for all s; by the definition (3.1) of the Zak transform we easily check that this is equivalent to the requirement that $\sum_k|\phi(2s-2k)|^2$ be bounded for all s, which is certainly true if, as in Proposition 2.1, ϕ decays faster than $|\xi|^{-1}$. Note that we have used $\Phi(-t,s)=\overline{\Phi(t,s)}$, which is true for real functions ϕ. All this proves the following proposition.

PROPOSITION 3.1. *Let ϕ be as in Proposition* 2.1. *Then* (2.5) *is satisfied, i.e.,*

$$\sum_{\ell\in\mathbb{Z}}\phi(\xi+\ell)\phi(\xi+\ell+2j)=\delta_{j0}$$

if and only if the Zak transform $\Phi=U_Z\phi$ of ϕ, as defined by (3.1) *satisfies*

$$|\Phi(t,s)|^2+|\Phi(t,s+\tfrac{1}{2})|^2=2 \tag{3.6}$$

for almost all $t,s\in[0,1]^2$.

4. Constructing solutions. Now that we have reduced the infinitely many conditions (2.5) to the single condition (3.6), we can get down to the business of constructing explicit "nice" ϕ satisfying (2.5). Typically, we start with a real function g with exponential decay,

$$|g(x)| \leqq C e^{-\lambda|x|}, \tag{4.1}$$

such that its Fourier transform has exponential decay as well,

$$|\hat{g}(y)| \leqq C e^{-\mu|y|}. \tag{4.2}$$

Define $G = U_Z g$; G is well defined and continuous. Since g is real, we have, for all $t, s \in \mathbb{R}$,

$$G(-t, s) = \overline{G(t, s)}. \tag{4.3}$$

Assume that

$$\inf_{t,s \in [0,1]} [|G(t, s)|^2 + |G(t, s+\tfrac{1}{2})|^2] > 0. \tag{4.4}$$

We then define

$$\phi = U_Z^{-1}\Phi, \tag{4.5}$$

where

$$\Phi(t, s) = \sqrt{2}\,\frac{G(t, s)}{[|G(t, s)|^2 + |G(t, s+\frac{1}{2})|^2]^{1/2}}. \tag{4.6}$$

Then the following theorem holds.

THEOREM 4.1. *The function ϕ, defined by* (4.5), *is a real function, and satisfies* (2.5). *Furthermore, both ϕ and $\hat{\phi}$ have exponential decay.*

Proof. 1. It follows from (4.3) and (4.6) that

$$\Phi(-t, s) = \overline{\Phi(t, s)},$$

so that, using (3.4) and (3.2),

$$\begin{aligned}\overline{\phi(x)} &= \frac{1}{\sqrt{2}}\int_0^1 dt\, \overline{\Phi\left(t, \frac{x}{2}\right)} = \frac{1}{\sqrt{2}}\int_0^1 dt\, \Phi\left(-t, \frac{x}{2}\right)\\ &= \frac{1}{\sqrt{2}}\int_{-1}^0 dt\, \Phi\left(t, \frac{x}{2}\right) = \frac{1}{\sqrt{2}}\int_0^1 dt\, \Phi\left(t, \frac{x}{2}\right) = \phi(x).\end{aligned}$$

2. To prove that ϕ has exponential decay, we first extend the definition domain of G from $\mathbb{R}^2$ to $(\mathbb{R} + i(-\lambda/\pi, \infty)) \times \mathbb{R}$. From (4.1) we see that the series

$$G(t+i\tau, s) = \sqrt{2} \sum_{\ell \in \mathbb{Z}} e^{2\pi i(t+i\tau)\ell} g(2(s-l)) \tag{4.7}$$

converges absolutely for $\tau > -\lambda/\pi$. The function $G(z, s)$ is continuous on $(\mathbb{R} + i(-\lambda/\pi, \infty)) \times \mathbb{R}$, and $G(\cdot, s)$ is analytic on $\mathbb{R} + i(-\lambda/\pi, \infty)$ for every $s \in \mathbb{R}$. Moreover,

$$\begin{aligned} G(z, s+1) &= e^{2\pi i z} G(z, s),\\ G(z+1, s) &= G(z, s). \end{aligned} \tag{4.8}$$

We also define, for $z \in \mathbb{R} + i(-\lambda/\pi, \infty)$, $s \in \mathbb{R}$

$$\mathcal{G}(z, s) = G(z, s)G(-z, s) + G(z, s+\tfrac{1}{2})G(-z, s+\tfrac{1}{2}). \tag{4.9}$$

Then $\mathcal{G}(\cdot, s)$ is analytic on $\mathbb{R} + i(-\lambda/\pi, \infty)$ for every $s \in \mathbb{R}$, and

$$\mathcal{G}(z+1, s) = \mathcal{G}(z, s) = \mathcal{G}(z, s+\tfrac{1}{2}) \tag{4.10}$$

for all $z \in \mathbb{R}+i(-\lambda/\pi, \infty)$, $s \in \mathbb{R}$. Using (4.2), we can show that G is uniformly continuous on $(\mathbb{R}+i[-\lambda/\pi, \infty)) \times [0, 1]$; together with (4.10) this implies that $\mathcal{G}$ is uniformly continuous on $(\mathbb{R}+i[-\lambda/\pi, \infty)) \times \mathbb{R}$. On the other hand, the restriction of $\mathcal{G}$ to $\mathbb{R} \times \mathbb{R}$ is real, and bounded below away from zero by (4.4). It follows that there exists $\tilde{\lambda} > 0$ so that $|\mathcal{G}|$ is bounded below away from zero on $(\mathbb{R}+i[-\tilde{\lambda}, \tilde{\lambda}]) \times \mathbb{R}$. We can therefore define $\mathcal{G}^{-1/2}$ as a uniformly continuous function on $(\mathbb{R}+i[-\tilde{\lambda}, \tilde{\lambda}]) \times \mathbb{R}$; $\mathcal{G}(z, s)^{-1/2}$ is analytic in $z \in \mathbb{R}+i(-\tilde{\lambda}, \tilde{\lambda})$ for all $s \in \mathbb{R}$. We can therefore extend (4.6), and define for $z \in \mathbb{R}+i(-\tilde{\lambda}, \tilde{\lambda})$, $s \in \mathbb{R}$,

$$\Phi(z, s) = \sqrt{2}\, \mathcal{G}(z, s)^{-1/2} G(z, s).$$

By (4.8) and (4.10) this extension satisfies

$$\begin{aligned} \Phi(z+1, s) &= \Phi(z, s), \\ \Phi(z, s+1) &= e^{2\pi i z} \Phi(z, s). \end{aligned} \tag{4.11}$$

We can now use this extension to prove exponential decay of ϕ. By (4.5) and (3.4)

$$\phi(x) = \frac{1}{\sqrt{2}} \int_0^1 dt\, \Phi\left(t, \frac{x}{2}\right).$$

Assume that $x \geqq 0$. (We will treat $x \leqq 0$ afterwards.) Using the analyticity of Φ in $t+i\tau$, we can deform the integration path,

$$\phi(x) = \frac{1}{\sqrt{2}} \left[\int_0^\Lambda d\tau\, \Phi\left(i\tau, \frac{x}{2}\right) + \int_0^1 dt\, \Phi\left(t+i\Lambda, \frac{x}{2}\right) + \int_\Lambda^0 d\tau\, \Phi\left(1+i\tau, \frac{x}{2}\right) \right],$$

where we assume $0 < \Lambda < \tilde{\lambda}$. Since $\Phi(1+i\tau, x/2) = \Phi(i\tau, x/2)$, the first and third integral cancel out. If $x = 2n + 2x_1$, with $x_1 \in [0, 1]$, then, by (4.11),

$$\begin{aligned} |\phi(x)| &= \frac{1}{\sqrt{2}} \left| \int_0^1 dt\, e^{2\pi i n(t+i\Lambda)} \Phi(t+i\Lambda, x_1) \right| \\ &\leqq \frac{1}{\sqrt{2}} e^{-2\pi\Lambda n} \sup_{\substack{z \in [0,1]+i[-\tilde{\lambda}, \tilde{\lambda}] \\ s \in [0,1]}} |\Phi(z, s)| \\ &\leqq C'\, e^{-\pi\Lambda x}. \end{aligned}$$

For $x \leqq 0$ we use the same argument, but we deform the integration path by going into the $\operatorname{Im} z < 0$ half plane. It follows that for all Λ such that

$$\Lambda < \min\left(\frac{\lambda}{\pi}, \inf\{|\tau|;\ \mathcal{G}(t+i\tau, s) = 0 \text{ for some } t, s \in [0, 1]\}\right),$$

there exists a constant C_Λ such that

$$|\phi(x)| \leqq C_\Lambda\, e^{-\pi\Lambda|x|}. \tag{4.12}$$

3. To prove the exponential decay of $\hat{\phi}$, we use the connection (3.5) between the Zak transforms of a function and of its Fourier transform. Because of (4.2) and (3.5b), arguments similar to those in step 2 above show that G can be extended to a uniformly continuous function on $\mathbb{R} \times (\mathbb{R}+i(\mu/4\pi, \infty))$, and that, for every $t \in \mathbb{R}$, $G(t, s+i\sigma)$ is analytic in $s+i\sigma \in \mathbb{R}+i(\mu/4\pi, \infty)$. We can now define, for $t \in \mathbb{R}$, $w = s+i\sigma \in \mathbb{R}+i(\mu/4\pi, \infty)$,

$$\Gamma(t, w) = G(t, w) G(-t, w) + G(t, w+\tfrac{1}{2}) G(-t, w+\tfrac{1}{2}).$$

Again $\Gamma(t, w)$ is analytic, and there exists $\tilde{\mu} > 0$ so that $|\Gamma|$ is bounded below away from zero on $\mathbb{R} \times (\mathbb{R} + i[-\tilde{\mu}, \tilde{\mu}])$. It follows that Φ has an extension to $\mathbb{R} \times (\mathbb{R} + i[-\tilde{\mu}, \tilde{\mu}])$,

$$\Phi(t, s+i\sigma) = \sqrt{2}\, G(t, s+i\sigma)\Gamma(t, s+i\sigma)^{-1/2},$$

which is analytic in $s + i\sigma$ for every fixed t, and which satisfies

$$\Phi(t, w+1) = e^{2\pi i t}\Phi(t, w),$$

$$\Phi(t+1, w) = \Phi(t, w).$$

By (3.4) and (3.5a) we have

$$\begin{aligned}\hat{\phi}(y) &= \frac{1}{\sqrt{2}} \int_0^1 ds\, (U_Z\hat{\phi})\left(s, \frac{y}{2}\right) \\ &= \frac{1}{2\sqrt{2}} \sum_{j=0}^{3} \int_0^1 ds\, e^{\pi i y(s+j)} \Phi\left(-2y, \frac{s+j}{4}\right).\end{aligned}$$

We can now play the same game as before (deform the integral over s into the complex plane, $\cdots$). The result is that for all Δ such that

$$\Delta < \min\left(\frac{\mu}{\pi}, 4 \inf\{|\sigma|;\ \Gamma(t, s+i\sigma) = 0 \text{ for some } t, s \in [0, 1]\}\right),$$

there exists a constant $\hat{C}_\Delta$ such that

$$|\hat{\phi}(y)| \leqq \hat{C}_\Delta\, e^{-\pi\Delta|y|}. \tag{4.13}$$

4. It remains to show that ϕ satisfies (2.5). It is obvious from $|G(t, s+1)| = |G(t, s)|$ and from (4.6) that

$$|\Phi(t, s)|^2 + |\Phi(t, s+\tfrac{1}{2})|^2 = 2 \tag{4.14}$$

for all $t, s \in \mathbb{R}$. Because of the exponential decay of ϕ and $\hat{\phi}$, all the manipulations of § 3 are indeed allowed, and (4.14) implies (2.5). $\square$

Any function g satisfying (4.1), (4.2), and (4.4) can therefore be used to construct an orthonormal Wilson basis of type (1.9). An explicit example is given by the Gaussian

$$g(x) = (2\nu)^{1/4}\, e^{-\nu\pi x^2}. \tag{4.15}$$

The Zak transform of g is related to one of Jacobi's theta functions,

$$\begin{aligned}G(t, s) &= \sqrt{2}\,(2\nu)^{1/4}\, e^{-4\nu\pi s^2} \sum_{\ell} e^{-4\nu\pi\ell^2}\, e^{2\pi\ell(4\nu s + it)} \\ &= \sqrt{2}\,(2\nu)^{1/4}\, e^{-4\nu\pi s^2}\theta_3(t - 4i\nu s\,|\,4i\nu),\end{aligned} \tag{4.16}$$

with Bateman's notation [26]

$$\theta_3(z\,|\,\tau) = 1 + 2\sum_{\ell=1}^{\infty} \cos(2\pi\ell z)\, e^{i\pi\tau\ell^2}.$$

As defined by (4.16), the function G has only one zero in $[0, 1]^2$, namely in $t = s = \frac{1}{2}$ [26]. Consequently, (4.4) is satisfied. Since g and $\hat{g}(y) = (2/\nu)^{1/4}\, e^{-\pi y^2/\nu}$ obviously have exponential decay, the construction (4.5)-(4.6) does lead to a Wilson basis with exponential phase space localization. For $\nu = 0.5$ we find

$$\inf\{|\tau|;\ \mathcal{G}(t+i\tau, s) = 0 \text{ for some } t, s \in [0, 1]\} = 0.5,$$

$$\inf\{|\sigma|;\ \Gamma(t, s+i\sigma) = 0 \text{ for some } t, s \in [0, 1]\} = 0.25.$$

Consequently, for every $\varepsilon > 0$ there exists C_ε such that

$$\begin{aligned} &|\phi(x)| \leqq C_\varepsilon\, e^{-(\pi-\varepsilon)|x|/2}, \\ &|\hat{\phi}(y)| \leqq C_\varepsilon\, e^{-(\pi-\varepsilon)|x|}. \end{aligned} \tag{4.17}$$

Remarks. 1. The decay rates in (4.17) can be adjusted by starting with a Gaussian different from (4.15). For $\nu = 2^{-1/2}$ e.g., we find that the corresponding ϕ and $\hat{\phi}$ are bounded by

$$\begin{aligned} &|\phi(x)| \leqq C_\varepsilon \exp(-(\pi-\varepsilon)|x|/\sqrt{2}), \\ &|\hat{\phi}(y)| \leqq C_\varepsilon \exp(-(\pi-\varepsilon)|y|/\sqrt{2}). \end{aligned} \tag{4.18}$$

2. It is easy to show that if g is an even function, then ϕ is even as well.

3. In [16] the explanation for the existence and exponential decay of the basis constructed by Wilson in [15] starts from an ansatz different from (1.9): the bimodal functions used as a starting point are of the form

$$g\left(\xi - \frac{2m+1}{4}\right) + (-1)^m g\left(\xi + \frac{2m+1}{4}\right). \tag{4.19}$$

For this ansatz the normalization (2.1) and the "completeness requirement" (2.2) do not reduce to the same condition. The orthonormalization of the functions in (4.19), starting from a "nice" g, results therefore in

$$\hat{f}_m(\xi) = \phi_m^1\left(\xi - \frac{2m+1}{4}\right) + \phi_m^2\left(\xi + \frac{2m+1}{4}\right),$$

where the $\phi_m^{1,2}$ depend on m. In the orthonormalization procedure in [16] the "overlap matrix" of the functions (4.19) is used. This overlap matrix also contains the quantity $|G(t,s)|^2 + |G(t, s+\frac{1}{2})|^2$, where G is the Zak transform of g (see Appendix B in [16]; the notation is very different, however). The merit of the present construction, starting from (1.9), is that the orthonormality (2.1) automatically follows once (2.2) is established; moreover, (2.1) + (2.2) are equivalent to the single condition (3.6), which enables us to construct, via (4.5)-(4.6) a *single* function ϕ generating the whole Wilson basis.

5. The link with tight frames. We start by briefly reviewing some material concerning "frames." Frames were introduced by Duffin and Schaeffer [27] in the context of nonharmonic Fourier series; in [23] and [7] special frames, constituted by families of functions of type (1.1), were studied in connection with the windowed Fourier transform. We review here some results from [7].

A family of g_{mn}, as defined in (1.1), constitutes a frame if there exist $A > 0$, $B < \infty$ such that, for all f in $L^2(\mathbb{R})$,

$$A\|f\|^2 \leqq \sum_{m,n} |\langle g_{mn}, f\rangle|^2 \leqq B\|f\|^2. \tag{5.1}$$

This condition can also be rewritten as

$$A\,\mathrm{Id} \leqq \mathbf{P} \leqq B\,\mathrm{Id}, \tag{5.2}$$

where $\mathbf{P}$ is the positive operator

$$\mathbf{P} = \sum_{m,n} P_{mn}, \qquad P_{mn}f = \langle g_{mn}, f\rangle g_{mn}. \tag{5.3}$$

If the g_{mn} constitute a frame, then functions $f \in L^2(\mathbb{R})$ can be completely characterized by the family of inner products $(\langle g_{mn}, f\rangle)_{m,n\in\mathbb{Z}}$, and there exists a numerically stable inversion procedure to reconstruct f from these inner products,

$$f = \sum_{m,n} \langle g_{mn}, f\rangle \tilde{g}_{mn},$$

with

$$\tilde{g}_{mn}(x) = e^{2\pi i\alpha mn}\tilde{g}(x-\beta n),$$
$$\tilde{g} = \mathbf{P}^{-1}g,$$

with $\mathbf{P}$ as defined by (5.3). Because of (5.2), $\mathbf{P}$ has a bounded inverse, so that $\tilde{g}$ is well defined. A special case arises when the frame is *tight*, i.e., when the frame bounds A and B are equal,

$$\sum_{m,n\in\mathbb{Z}} |\langle g_{mn}, f\rangle|^2 = A\|f\|^2.$$

It then follows that

$$\mathbf{P} = A\,\mathrm{Id},$$
$$\tilde{g} = A^{-1}g,$$
$$f = A^{-1}\sum_{m,n\in\mathbb{Z}} \langle g_{mn}, f\rangle g_{mn}.$$

In general, frames are redundant (they contain "too many" vectors, or more precisely, any frame vector lies in the closed linear span of all the others). If the frame is tight, then A indicates how redundant the frame is; for tight frames of type (1.1) we find [7]

$$A = (\alpha\beta)^{-1}\|g\|^2. \tag{5.4}$$

A frame of type (1.1) can only be an orthonormal basis if $\alpha\beta = 1$ (and if, moreover, g is chosen appropriately), corresponding to $A = 1$, or no redundancy. Tight frames with "nice" g exist if and only if $\alpha\beta < 1$; see [23] for a construction with compactly supported g.

Let us now specialize to the case $\alpha = .5$, $\beta = 1$,

$$g_{mn}(x) = e^{im\pi x}g(x-n).$$

The density of the phase space lattice corresponding to the g_{mn} is then twice as high as for an orthonormal basis. Suppose g is "nice," i.e., both g and $\hat{g}$ have fast decay at ∞. Let us investigate under which conditions on g the g_{mn} constitute a frame (respectively, tight frame). Because $(\alpha\beta)^{-1} = 2$ is an integer, the Zak transform is a natural tool to study these questions, as observed in [8]. Using (3.1), we find

$$(U_Z g_{m2n})(t,s) = e^{-2\pi itn}\, e^{2\pi ims}G(t,s),$$
$$(U_Z g_{m2n-1})(t,s) = e^{-2\pi itn}\, e^{2\pi ims}G(t, s+\tfrac{1}{2}),$$

where $G = U_Z g$. It follows that, for all $h_1, h_2 \in L^2(\mathbb{R})$,

$$\begin{aligned}\sum_{m,n\in\mathbb{Z}} \langle h_1, P_{m2n}h_2\rangle &= \sum_{m,n\in\mathbb{Z}} \langle h_1, g_{m2n}\rangle\langle g_{m2n}, h_2\rangle\\ &= \sum_{m,n\in\mathbb{Z}} \left[\int_0^1 dt\int_0^1 ds\, U_Z h_1(t,s)\overline{G(t,s)}\, e^{2\pi itn}\, e^{-2\pi ims}\right]^*\\ &\quad\cdot\left[\int_0^1 dt\int_0^1 ds\, U_Z h_2(t,s)\overline{G(t,s)}\, e^{2\pi itn}\, e^{-2\pi ims}\right]\\ &= \int_0^1 dt\int_0^1 ds\,\overline{U_Z h_1(t,s)}\, U_Z h_2(t,s)|G(t,s)|^2.\end{aligned}$$

Consequently, $U_Z[\sum_{m,n\in\mathbb{Z}} P_{m2n}]U_Z^{-1}$ is multiplication by $|G(t,s)|^2$ in $\mathscr{Z}$. Similarly, $U_Z[\sum_{m,n\in\mathbb{Z}} P_{m2n-1}]U_Z^{-1}$ is multiplication by $|G(t,s+\frac{1}{2})|^2$. Consequently, $\mathbf{P}=\sum_{m,n} P_{mn}$ is unitarily equivalent to multiplication by $|G(t,s)|^2+|G(t,s+\frac{1}{2})|^2$ on $\mathscr{Z}$. It follows that the g_{mn} constitute a frame, or equivalently that $\mathbf{P}$ satisfies (5.2), if and only if

$$0<A\leqq|G(t,s)|^2+|G(t,s+\tfrac{1}{2})|^2\leqq B<\infty$$

for all $t, s\in[0,1]$. All this is summarized in the following proposition.

PROPOSITION 5.1. *The functions* $g_{mn}(x)=e^{im\pi x}g(x-n)$ *constitute a frame if and only if the Zak transform* $G=U_Zg$ *of* g, *as defined by* (3.1), *satisfies*

$$A=\inf_{t,s\in[0,1]}[|G(t,s)|^2+|G(t,s+\tfrac{1}{2})|^2]>0$$

and

$$B=\sup_{t,s\in[0,1]}[|G(t,s)|^2+|G(t,s+\tfrac{1}{2})|^2]<\infty.$$

Note that if $|g(x)|\leqq C(1+|x|)^{-1-\varepsilon}$, then G is bounded, and B is automatically finite. There are other procedures than the Zak transform to check whether the g_{mn} constitute a frame [7]. The point of Proposition 5.1 is that any reasonably well-localized g such that the g_{mn} constitute a frame can be used as a starting point in the construction of ϕ in § 4. Note that the computations above also prove the following proposition.

PROPOSITION 5.2. *Let* ϕ *be a real function such that* $|\phi(x)|\leqq C(1+|x|)^{-1-\varepsilon}$ *and* $\int dx\,|\phi(x)|^2=1$. *Then the following are equivalent*:

(1) *The* ψ_{mn}, *as defined by* (1.9), *constitute an orthonormal basis,*

(2) *The Zak transform* $\Phi=U_Z\phi$ *of* ϕ *satisfies*

$$|\Phi(t,s)|^2+|\Phi(t,s+\tfrac{1}{2})|^2=2,$$

(3) *The functions* $\phi_{mn}(x)=e^{im\pi x}\phi(x-n)$, $m, n\in\mathbb{Z}$, *constitute a tight frame.*

Proof.

(1) $\Leftrightarrow$ (2) is proved in Proposition 5.2.

Define now $\mathbf{P}(\phi)$ by

$$\mathbf{P}(\phi)f=\sum_{m,n}\langle\phi_{mn},f\rangle\phi_{mn}.$$

Then, by the computation above,

$$\mathbf{P}(\phi)=U_Z^{-1}\{\text{multiplication with }[|\Phi(t,s)^2|+|\Phi(t,s+\tfrac{1}{2})|^2]\}U_Z. \tag{5.5}$$

If (2) holds, then it follows that $\mathbf{P}(\phi)=2\,\mathrm{Id}$, i.e.,

$$\sum_{m,n}|\langle\phi_{mn},f\rangle|^2=2\|f\|^2,$$

so the ϕ_{mn} constitute a tight frame.

On the other hand, if the ϕ_{mn} constitute a tight frame, i.e.,

$$\sum_{m,n}|\langle\phi_{mn},f\rangle|^2=A\|f\|^2,$$

then $A=2$ by (5.4) ($\alpha=.5$, $\beta=1$, and $\|g\|=1$). It follows that $\mathbf{P}(\phi)=2\,\mathrm{Id}$, which by (5.5) implies (2). □

Remark. From this analysis it follows that the construction in § 4 and § 6 below can also be used to generate tight frames with exponential localization in both time and frequency. The construction in § 6 can easily be extended to tight frames with arbitrary redundancy. These tight frames contrast with those constructed in [23], where either ϕ or $\hat{\phi}$ had compact support.

Proposition 5.2 leads to the following interpretation of our Wilson bases. Suppose that the ϕ_{mn} constitute a tight frame. Since $\alpha\beta = .5$, this tight frame has redundancy 2, i.e., it has "two times as many vectors" as an orthonormal basis. The Wilson basis vectors generated by ϕ via (1.9) are given by

$$(\psi_{mn})\hat{}(\xi) = e^{2\pi i n\xi}\hat{f}_m(\xi),$$

or

$$(\psi_{1n})\hat{}(\xi) = e^{2\pi i n\xi}\phi(\xi) = \phi_{02n}(\xi), \tag{5.6a}$$

$$(\psi_{2\ell+\kappa n})\hat{}(\xi) = \frac{1}{\sqrt{2}}\, e^{2\pi i n\xi}\, e^{i\pi\kappa\xi}[\phi(\xi-\ell)+(-1)^{\ell+\kappa}\phi(\xi+\ell)]$$

$$= \frac{1}{\sqrt{2}}(\phi_{\ell 2n+\kappa}+(-1)^{\ell+\kappa}\phi_{-\ell 2n+\kappa})(\xi), \tag{5.6b}$$

$$\ell \in \mathbb{N}\backslash\{0\}, \qquad \kappa = 0 \text{ or } 1.$$

Formula (1.9) can therefore be viewed as a procedure eliminating the redundancy factor 2 from the tight frame ϕ_{mn} by choosing only the ϕ_{0n} with even n, and replacing every pair $\phi_{\ell n}$, $\phi_{-\ell n}$ ($\ell \neq 0$) by one judiciously chosen linear combination of these two vectors. It seems a small miracle that the result is an orthonormal basis!

Note that (5.6) can be made even simpler by a relabelling of the ψ_{mn}. Denote

$$\Psi_{\ell 2n+\kappa} = \psi_{2\ell+\kappa n}, \qquad \ell \neq 0, \quad \kappa = 0 \text{ or } 1,$$

$$\Psi_{0n} = \psi_{1n}.$$

Then (5.6) becomes

$$(\Psi_{0n})\hat{} = \phi_{02n},$$

$$(\Psi_{\ell n})\hat{} = \frac{1}{\sqrt{2}}(\phi_{\ell n}+(-1)^{\ell+n}\phi_{-\ell n}), \qquad \ell \neq 0,$$

making the reduction from the right frame with redundancy 2 to the orthonormal basis even more elegant.

6. The construction revisited. The equivalence between (4.4) and the frame condition (5.1) leads to an alternate construction for the function ϕ which is very easy to implement numerically.

Choose g such that (4.1), (4.2), and (4.4) are satisfied. Then, by the argument in § 5,

$$U_Z \mathbf{P} U_Z^{-1} = \text{multiplication by } |G(t,s)|^2 + |G(t, s+\tfrac{1}{2})|^2;$$

consequently,

$$\phi = U_Z^{-1}\frac{\sqrt{2}\,G}{\sqrt{|G(t,s)|^2+|G(t,s+\frac{1}{2})|^2}} = U_Z^{-1}U_Z\mathbf{P}^{-1/2}U_Z^{-1}\sqrt{2}\,G$$

$$= \sqrt{2}\,\mathbf{P}^{-1/2}U_Z^{-1}G = \sqrt{2}\,\mathbf{P}^{-1/2}g.$$

The operator $\mathbf{P}^{-1/2}$ can be written as a convergent series. Since

$$A\,\mathrm{Id} \leqq \mathbf{P} \leqq B\,\mathrm{Id}$$

for all $A>0$, $B<\infty$ satisfying

$$A \leqq \inf_{t,s\in[0,1]} [|G(t,s)|^2+|G(t,s+\tfrac{1}{2})|^2],$$

$$B \geqq \sup_{t,s\in[0,1]} [|G(t,s)|^2+|G(t,s+\tfrac{1}{2})|^2],$$

we have

$$\begin{aligned} \mathbf{P}^{-1/2} &= \left(\frac{2}{A+B}\right)^{1/2}\left[\mathrm{Id}-\left(\mathrm{Id}-\frac{2\mathbf{P}}{A+B}\right)\right]^{-1/2} \\ &= \left(\frac{2}{A+B}\right)^{1/2} \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2}\left(\mathrm{Id}-\frac{2\mathbf{P}}{A+B}\right)^k, \end{aligned} \tag{6.1}$$

where the series converges because

$$\left\|\mathrm{Id}-\frac{2\mathbf{P}}{A+B}\right\| \leqq \frac{B-A}{B+A}<1.$$

This can be used to write ϕ as a combination of g_{mn}, with coefficients computed recursively. For instance, if g is Gaussian, $g^\nu(x)=(2\nu)^{1/4}\, e^{-\nu\pi x^2}$, then we find

$$\phi(x)=\frac{2}{\sqrt{A_\nu+B_\nu}} \sum_{m,n\in\mathbb{Z}} a_{mn} g^\nu_{mn}(x) \tag{6.2}$$

with

$$\begin{aligned} a_{mn} &= \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2} b^k_{mn}, \\ b^k_{mn} &= \left(1-\frac{2}{A_\nu+B_\nu}\right) b^{k-1}_{mn} - \frac{2}{A_\nu+B_\nu} \sum_{(m',n')\neq(m,n)} \omega_{mn,m'n'} b^{k-1}_{m'n'}, \end{aligned} \tag{6.3}$$

where

$$\omega_{mn,m'n'} = \exp\left[i(m'-m)(n+n')\frac{\pi}{2}-\frac{\nu\pi}{2}(n-n')^2-\frac{\pi}{8\nu}(m-m')^2\right],$$

$$b^0_{mn}=\delta_{m0}\delta_{n0}.$$

While this seems lengthy, it is very easy to program on a computer. The procedure converges at least as fast as a geometric series in $(B_\nu-A_\nu)/(B_\nu+A_\nu)$. For $\nu=.5$ we find $A_\nu=1.670$, $B_\nu=2.361$, $(B_\nu-A_\nu)/(B_\nu+A_\nu)=.1712$; for $\nu=2^{-1/2}$, we have $A_\nu=1.533$, $B_\nu=2.492$, $(B_\nu-A_\nu)/(B_\nu+A_\nu)=.2381$. Figures 1 and 2 give graphs of ϕ and $\hat{\phi}$, for $\nu=.5$ and $\nu=2^{-1/2}$, respectively.

Remarks. 1. A and B can be computed via the Zak transform:

$$A=\inf_{t,s\in[0,1]} [|(U_Z g)(t,s)|^2+|(U_Z g)(t,s+\tfrac{1}{2})|^2],$$

$$B=\sup_{t,s\in[0,1]} [|(U_Z g)(t,s)|^2+|(U_Z g)(t,s+\tfrac{1}{2})|^2].$$

In [7] an alternative way of estimating A and B is given, leading to a lower bound for A and an upper bound for B, without recourse to the Zak transform. Using the Poisson summation formula, we find

$$\mathbf{m}(g)-\mathbf{r}(g) \leqq A \leqq B \leqq \mathbf{M}(g)+\mathbf{r}(g), \tag{6.4}$$

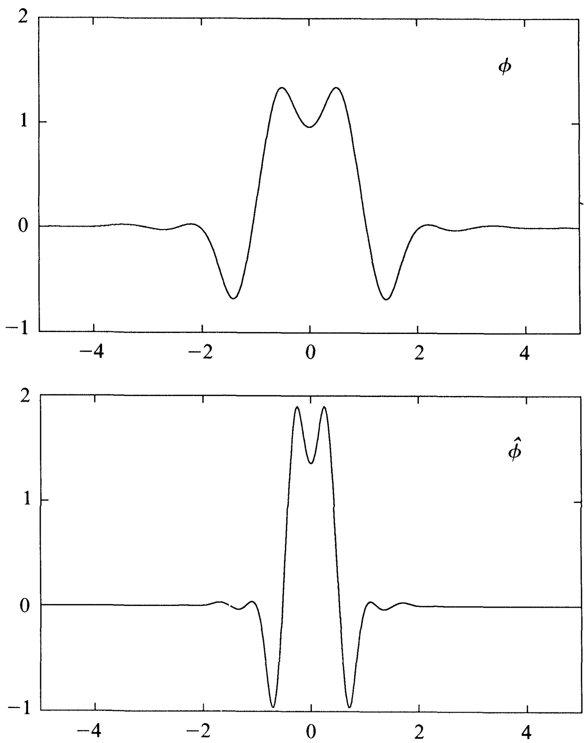

FIG. 1. *Plots of ϕ and $\hat{\phi}$, constructed from $g(x)=e^{-\pi x^2/2}$. For this choice of g, we have $\hat{\phi}(y)=\sqrt{2}\,\phi(2y)$ (see (6.6)). To draw these graphs, the recursive computation (6.2), (6.3) was used.*

where

$$\mathbf{m}(g)=\inf_{x\in[0,1]}\sum_n |g(x-n)|^2,$$

$$\mathbf{M}(g)=\sup_{x\in[0,1]}\sum_n |g(x-n)|^2,$$

$$\mathbf{r}(g)=2\sum_{k=1}^{\infty}[\beta(2k)\beta(-2k)]^{1/2},$$

$$\beta(s)=\sup_{x\in[0,1]}\sum_n |g(x-n)g(x-n+s)|.$$

Note that the lower bound in (6.4) also gives a sufficient condition ensuring that (4.4) holds, without having to check the Zak transform. In some cases, more efficient bounds can be computed from the Fourier transform $\hat{g}$ of g. We obtain [7]

$$\tilde{\mathbf{m}}(g)-\tilde{\mathbf{r}}(g)\leqq A\leqq B\leqq \tilde{\mathbf{M}}(g)-\tilde{\mathbf{r}}(g)$$

with

$$\tilde{\mathbf{m}}(g)=\inf_{\xi\in[0,1/2]}\sum_n |\hat{g}(\xi-n/2)|^2,$$

$$\tilde{\mathbf{M}}(g)=\sup_{\xi\in[0,1/2]}\sum_n |\hat{g}(\xi-n/2)|^2,$$

$$\tilde{\mathbf{r}}(g)=2\sum_{k=1}^{\infty}[\tilde{\beta}(k)\tilde{\beta}(-k)]^{1/2},$$

$$\tilde{\beta}(s)=\sup_{\xi\in[0,1/2]}\sum_n |\hat{g}(\xi-n/2)\hat{g}(\xi-n/2+s)|.$$

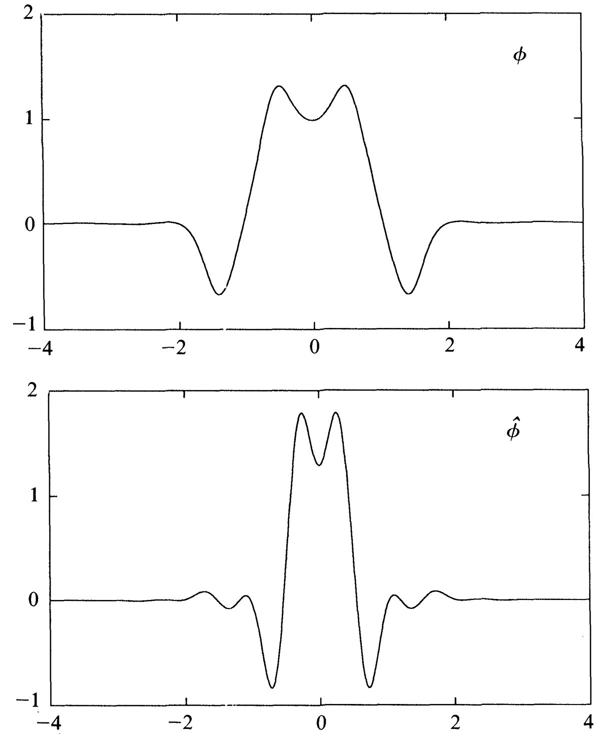

FIG. 2. *Plots of* ϕ *and* $\hat{\phi}$ *constructed from* $g(x)=2^{1/8}\, e^{-\pi x^2/\sqrt{2}}$. *For this choice of* g, *the decay rates of* ϕ *and* $\hat{\phi}$ *are identical (see* (4.18)). *We have again used* (6.2), (6.3) *to compute* ϕ; $\hat{\phi}$ *is simply obtained by replacing* g^{ν}_{mn} *in* (6.2) *by* $(g^{\nu}_{mn})^{\wedge}(y)=(-1)^{mn}\, e^{2\pi i y n} g^{1/\nu}(y+(m/2))$.

2. Let us introduce the notation F for the Fourier transform and D_a for the dilations $(D_a f)(x)=|a|^{1/2} f(ax)$, and let us write $\mathbf{P}(g)$, $\phi(g)$ to make the dependence of the operator $\mathbf{P}$ and the function ϕ on the function g more explicit. Then we easily check that

$$D_{1/2} F\mathbf{P}(g)=\mathbf{P}(D_{1/2}Fg)D_{1/2}F,$$

implying

$$D_{1/2}F\phi(g)=\phi(D_{1/2}Fg). \tag{6.5}$$

Denoting $\phi(g^{\nu})$ by ϕ_{ν}, where $g^{\nu}(x)=(2\nu)^{1/4}\exp(-\pi\nu x^2)$, we find therefore

$$(\phi_{\nu})^{\wedge}(y)=\sqrt{2}\,\phi_{(4\nu)^{-1}}(2y). \tag{6.6}$$

In particular, for $\nu=.5$, $(\phi_{1/2})^{\wedge}(y)=\sqrt{2}\,\phi_{1/2}(2y)$.

7. Conclusion. We have shown how to construct very simple Wilson bases ψ_{mn}, generated by a single function ϕ, via

$$\begin{aligned}
&\psi_{mn}(x)=f_m(x-n), \qquad n\in\mathbb{Z}, \quad m\in\mathbb{N}\backslash\{0\},\\
&\hat{f}_1(\xi)=\phi(\xi),\\
&\hat{f}_{2\ell+\kappa}(\xi)=\frac{1}{\sqrt{2}}[\phi(\xi-\ell)+(-1)^{\ell+\kappa}\phi(\xi+\ell)]\,e^{i\pi\kappa\xi}, \qquad \ell\in\mathbb{N}\backslash\{0\}, \quad \kappa=0 \text{ or } 1.
\end{aligned} \tag{7.1}$$

We have explicitly constructed such bases; in order to obtain exponential decay for both the ψ_{mn} and their Fourier transforms $\hat{\psi}_{mn}$, it suffices to choose a function g such that g and $\hat{g}$ have exponential decay, and such that condition (4.4) is satisfied, or equivalently, such that the $g_{mn}(x) = e^{\pi imx} g(x-n)$ constitute a frame. (For this it is sufficient that $\mathbf{m}(g) - \mathbf{r}(g) > 0$ or $\tilde{\mathbf{m}}(g) - \tilde{\mathbf{r}}(g) > 0$—see § 6.) The function ϕ can then be constructed from g either via the Zak transform (see § 4) or via a recursive algorithm (see § 6).

The functions f_m in (7.1) are given by the inverse Fourier transform of ϕ. If g is real and even, then so is ϕ, so that its Fourier transform and inverse Fourier transform coincide. We then have

$$f_1(x) = \hat{\phi}(x),$$

$$f_{2\ell+\kappa}(x) = \frac{1}{\sqrt{2}} \hat{\phi}\left(x + \frac{\kappa}{2}\right) e^{\pi i \ell \kappa} \left[e^{2\pi i \ell x} + (-1)^{\ell+\kappa} e^{-2\pi i \ell x} \right].$$

Using the relabelling

$$\Psi_{0n} = \psi_{1n}, \qquad \Psi_{\ell 2n+\kappa} = \psi_{2\ell+\kappa n}$$

we find

$$\Psi_{0n}(x) = \hat{\phi}(x-n),$$

$$\Psi_{\ell n}(x) = \sqrt{2}\, \hat{\phi}\left(x - \frac{n}{2}\right) \begin{cases} \cos(2\pi \ell x) & \text{if } \ell + n \text{ is even,} \\ \sin(2\pi \ell x) & \text{if } \ell + n \text{ is odd.} \end{cases}$$

It follows that the Wilson bases constructed here are very similar to the functions (1.1): the only difference is the alternate use of sines and cosines instead of complex exponentials. This trick is sufficient to beat the no-go Balian–Low theorem.

REFERENCES

[1] W. Kohn, *Analytic properties of Bloch waves and Wannier functions*, Phys. Rev., 115 (1959), pp. 809–821.

[2] D. Gabor, *Theory of communication*, J. Inst. Electr. Engrg. London (III), 93 (1946), pp. 429–457.

[3a] M. J. Bastiaans, *Gabor's signal expansion and degrees of freedom of a signal*, Proc. IEEE, 68 (1980), pp. 538–539.

[3b] ———, *A sampling theorem for the complex spectrogram and Gabor's expansion of a signal in Gaussian elementary signals*, Optical Engrg., 20 (1981), pp. 594–598.

[4a] A. J. E. M. Janssen, *Gabor representation of generalized functions*, J. Math. Appl., 80 (1981), pp. 377–394.

[4b] ———, *Gabor representation and Wigner distribution of signals*, Proc. IEEE (1984), pp. 41.B.2.1–41.B.2.4.

[5] V. Bargmann, P. Butera, L. Girardello, and J. R. Klauder, *On the completeness of coherent states*, Rep. Math. Phys., 2 (1971), pp. 221–228.

[6] A. M. Perelomov, *Note on the completeness of systems of coherent states*, Teor. i Matem. Fis., 6 (1971), pp. 213–224.

[7] I. Daubechies, *The wavelet transform, time-frequency localization and signal analysis*, IEEE Trans. Inf. Theory, 36 (1990) pp. 961–1005.

[8] I. Daubechies and A. Grossmann, *Frames of entire functions in the Bargmann space*, Comm. Pure Appl. Math., 41 (1988), pp. 151–164.

[9] R. Balian, *Un principe d'incertitude fort en théorie du signal ou en mécanique quantique*, C.R. Acad. Sci. Paris, Sér. 2, 292 (1981), pp. 1357–1361.

[10] F. Low, *Complete Sets of Wave-Packets*, in A Passion for Physics—Essays in Honor of Geoffrey Chew, World Scientific, Singapore, 1985, pp. 17–22.

[11] G. Battle, *Heisenberg proof of the Balian–Low theorem*, Lett. Math. Phys., 15 (1988), pp. 175–177.

[12] I. Daubechies and A. J. E. M. Janssen, *Two theorems on lattice expansions*, preprint.

[13] J. BOURGAIN, *A remark on the uncertainty principle for Hilbertian basis*, J. Funct. Anal., 79 (1988), pp. 136–143.

[14] T. STEGER, private communication, 1986.

[15] K. G. WILSON, *Generalized Wannier Functions*, Cornell University preprint, 1987.

[16] D. J. SULLIVAN, J. J. REHR, J. W. WILKINS, AND K. G. WILSON, *Phase Space Wannier Functions in Electronic Structure Calculations*, Cornell University, preprint, 1987.

[17] Y. MEYER, *Principe d'incertitude, bases hilbertiennes et algèbres d'opérateurs*, Séminaire Bourbaki, Paris, France, No. 662, 1985–1986.

[18] J. O. STROMBERG, *A modified Franklin system and higher order spline systems on $\mathbb{R}^n$ as unconditional bases for Hardy spaces*, Conf. in Honor of A. Zygmund, Vol. II, W. Beckner, A. P. Calderon, R. Fefferman, and P. Jones, eds., Wadsworth Math. Series, Belmont, CA, 1983, pp. 475–493.

[19] G. BATTLE, *A block spin construction of ondelettes. Part* I. *Lemarié functions*, Comm. Math. Phys., 110 (1987), pp. 601–615.

[20] P. G. LEMARIÉ, *Une nouvelle base d'ondelettes de $L^2(\mathbb{R}^n)$*, J. Math. Pures Appl., 67 (1988), pp. 227–236.

[21] I. DAUBECHIES, *Orthonormal basis of compactly supported wavelets*, Comm. Pure Appl. Math., 41 (1988), pp. 909–996.

[22] A. COHEN, private communication, 1988.

[23] I. DAUBECHIES, A. GROSSMANN, AND Y. MEYER, *Painless non-orthogonal expansions*, J. Math. Phys., 27 (1986), pp. 1271–1283.

[24a] J. ZAK, *Finite translations in solid state physics*, Phys. Rev. Lett., 19 (1967), pp. 1385–1397.

[24b] ———, *Dynamics of electrons in solids in external fields*, Phys. Rev., 168 (1968), pp. 686–695.

[24c] ———, *The kq representation in the dynamics of electrons in solids*, Solid State Physics, 27 (1972), pp. 1–62.

[25] A. J. E. M. JANSSEN, *The Zak transform: a signal transform for sampled time-continuous signals*, Philips J. Res., 43 (1988), pp. 23–69.

[26] BATEMAN PROJECT, *Higher Transcendental Functions, Vol.* 2, McGraw-Hill, New York, 1955.

[27] R. J. DUFFIN AND A. C. SCHAEFFER, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc., 72 (1952), pp. 341–366.