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TWO-SCALE DIFFERENCE EQUATIONS I. EXISTENCE AND GLOBAL REGULARITY OF SOLUTIONS*

INGRID DAUBECHIES†‡ AND JEFFREY C. LAGARIAS†

Abstract. A *two-scale difference equation* is a functional equation of the form $f(x)=\sum_{n=0}^{N} c_n f(\alpha x-\beta_n)$, where $\alpha>1$ and $\beta_0<\beta_1<\cdots<\beta_n$ are real constants, and c_n are complex constants. Solutions of such equations arise in spline theory, in interpolation schemes for constructing curves, in constructing wavelets of compact support, in constructing fractals, and in probability theory. This paper studies the existence and uniqueness of L^1-solutions to such equations. In particular, it characterizes L^1-solutions having compact support. A time-domain method is introduced for studying the special case of such equations where $\{\alpha, \beta_0, \cdots, \beta_n\}$ are integers, which are called *lattice two-scale difference equations.* It is shown that if a lattice two-scale difference equation has a compactly supported solution in $C^m(\mathbb{R})$, then $m<(\beta_n-\beta_0)/(\alpha-1)-1$.

Key words. wavelets, subdivision algorithms, fractals

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1. Introduction. A *two-scale difference equation* is a functional equation of the form

$$f(x)=\sum_{n=0}^{N} c_n f(\alpha x-\beta_n) \tag{1.1}$$

where $\alpha>1$ and $\beta_0<\beta_1<\cdots<\beta_n$ are real constants, and x takes real values, while the c_n are complex constants. The right side of (1.1) is typical for difference equations, and the name two-scale difference equation reflects the fact that (1.1) relates translates of scaled versions of the same function, involving two different scales. A *lattice two-scale difference equation* is the special case where α and all β_n are integers, i.e.,

$$f(x)=\sum_{n=0}^{N} c_n f(kx-n) \tag{1.2}$$

where $k\geqq 2$ is an integer. The apparently more general equation

$$f(x)=\sum_{n=-N_1}^{N_2} c_n f(kx-n) \tag{1.3}$$

can be reduced to the form (1.2) by the change of variable $y=x-N_1/(k-1)$.

This paper and its sequel (Daubechies and Lagarias (1988), hereafter called part II) study L^1-solutions of two-scale difference equations, and of lattice two-scale difference equations in particular. The basic questions concern the existence, uniqueness, and degree of regularity of solutions for a given equation. We treat in detail L^1-solutions having compact support. In fact, two-scale difference equations always have solutions in the sense of distributions and may also possess functions not in $L^1(\mathbb{R})$ as solutions, e.g., if $\sum_{n=0}^{N} c_n=1$, then the constant functions are solutions. However, only for special sets of $\{\alpha, \beta_n, c_n\}$ will (1.1) have any nonzero L^1-solutions.

Functions that satisfy lattice two-scale difference equations arise in several different contexts. G. de Rham is credited with an example of a continuous,

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† AT&T Bell Laboratories, Murray Hill, New Jersey 07974.

‡ "Bevoegdverklaard Navorser" at the Belgian National Foundation for Scientific Research (on leave); on leave also from the Department of Theoretical Physics, Vrije Universiteit Brussel, Belgium.

nowhere-differentiable function which satisfies (1.2) with $k=3$ and $c_0=1$, $c_1=c_{-1}=\frac{1}{3}$, $c_2=c_{-2}=\frac{2}{3}$, and all other $c_n\equiv 0$. (This was communicated to us by Meyer (1987). We have not found a direct reference to this function in de Rham's papers but similar functions appear in de Rham (1947), (1956), (1957), (1959).) Such functions also arise as limits of "uniform subdivision schemes" for constructing curves and surfaces. As observed and generalized in the work of Dahmen and Micchelli (1984), (1988) and Micchelli and Prautzsch (1987a), (1987b), (1989), normalized B-splines and d-dimensional box splines each satisfy a lattice two-scale difference equation with $k=2$. They point out that this two-scale property is really the basic ingredient in a subdivision algorithm for numerically evaluating B-spline curves and surfaces, given by Lane and Riesenfeld (1980). This can be exploited to define and study other subdivision schemes for the design of curves and surfaces, also characterized by a lattice two-scale equation (Cavaretta and Micchelli (1989)). Dyn and Levin (1989) similarly link subdivision algorithms with the study of a lattice two-scale equation. Dubuc (1986) proposed a dyadic interpolation scheme where the "fundamental function" satisfies an equation of type (1.3) with $k=2$. For special values of the parameters, he proved smoothness results of this fundamental function. Dyn, Gregory, and Levin (1987) independently and by different techniques proved similar results for the same dyadic interpolation schemes. In Deslauriers and Dubuc (1987) this interpolation scheme was applied to the construction of fractal objects and functions with fractal properties. Deslauriers and Dubuc (1989) extend the dyadic interpolation scheme to other integer values of k; they use the properties of solutions of (1.3) corresponding to specific values of the c_n to study Lagrange iterative interpolation processes. In another field, Daubechies (1988) constructed orthonormal bases of compactly supported wavelets, i.e., orthonormal bases $\{h_{mn}(x)\}$ of $L^2(\mathbb{R})$ generated by translating and dilating a single compactly supported function h via

$$h_{mn}(x)=2^{-m/2}h(2^{-m}x-n).$$

The construction of such h requires an auxiliary function which is a solution of a lattice two-scale difference equation, and our interest in these equations arose from these functions. All of these examples actually involve L^1-solutions having compact support.

Solutions of general two-scale difference equations (1.1) arise in other areas of mathematics as well. Kershner and Wintner (1935) considered symmetric Bernoulli convolutions $d\lambda(x,\beta)$ whose Fourier transform $\Lambda(u,\beta)=\int_{-\infty}^{\infty} e^{iux}\,d\lambda(x,\beta)$ has

$$\Lambda(u,\beta)=\prod_{n=0}^{\infty}\cos(\beta^n u). \tag{1.4}$$

For certain values of β in $(0,1)$ the measure $d\lambda(x,\beta)=\lambda'(x,\beta)\,dx$ is absolutely continuous, and $\lambda'(x,\beta)$ then satisfies the two-scale difference equation

$$\lambda'(x)=\frac{\beta}{2}\left(\lambda'\left(\frac{1}{\beta}x-1\right)+\lambda'\left(\frac{1}{\beta}x+1\right)\right).$$

Smoothness properties of this and related Bernoulli convolutions were studied by Jessen and Wintner (1935), Erdös (1939), (1940), Garsia (1962), and Brown and Moran (1973). It remains a difficult open problem to characterize the set of β for which $d\lambda(x,\beta)$ is absolutely continuous. More recently, Barnsley and Demko (1985, Ex. 21) in studying iterated function systems construct a function $f_\beta(z)$ defined on $\mathbb{C}-A$ where

A is the Cantor set, satisfying the two-scale difference equation

$$f_\beta(z)=\frac{3^\beta}{2}(f_\beta(3z-2)+f_\beta(3z)).$$

Here $f_\beta(z)=\int_A d\mu(x)/(z-x)^\beta$, where $d\mu$ is uniform measure on the Cantor set.

We use two methods for studying these equations. They are a Fourier transform approach that applies to the general equation (1.1), and a time-domain approach described further below that applies only to lattice two-scale difference equations. Part I describes Fourier transform results on existence and uniqueness, introduces the time-domain approach, and uses it to establish bounds on the smoothness of L^1-solutions of compact support. Part II studies the time-domain construction in detail and gives sufficient conditions for the existence of nonzero continuous solutions of compact support, and determines their local and global regularity properties.

The Fourier transform provides a method for the study of L^1-solutions f of general two-scale difference equations (1.1). The convolution character of the right side of (1.1) leads to an infinite product expansion for the Fourier transform $\hat{f}(u)$ permitting detailed study. Section 2 uses this approach to obtain existence and uniqueness results for L^1-solutions to (1.1). These depend in a crucial way on the quantity

$$\Delta=\alpha^{-1}\sum_{m=0}^{N} c_n. \tag{1.5}$$

There are no nonzero L^1-solutions if $|\Delta|<1$ or if $|\Delta|=1$ and $\Delta\neq 1$. The case of most interest is $\Delta=1$; it has at most one nonzero L^1-solution, up to a multiplicative scale factor. This solution, if it exists, is of compact support with support $(f)\subset[0,(\alpha-1)^{-1}N]$, and has $\int_{-\infty}^{\infty}f(x)\,dx\neq 0$. For $|\Delta|>1$ it is possible to have zero, one, or infinitely many L^1-solutions, which need not have compact support, depending on the values $\{\alpha,\beta_n,c_n\}$.

Section 3 studies L^1-functions of compact support solving (1.1), and shows that they are all derived from solutions of the case $\Delta=1$, in the following sense. If a two-scale difference equation (1.1) has a nonzero L^1-solution f of compact support, then it is unique (up to normalization), and necessarily,

(1) $\Delta=\alpha^m$ for some nonnegative integer m;

(2) The two-scale difference equation with $\Delta=1$ obtained by replacing the coefficients $\{c_n\}$ with $\{\alpha^{-m}c_n\}$ has a nonzero L^1-solution g of compact support, and for a suitable choice of normalization,

$$\frac{d^m}{dx^m}g(x)\equiv f(x)\quad\text{a.e;}$$

The remainder of part I and part II use time-domain methods that apply only to the special case of lattice two-scale difference equations (1.2). This approach exploits the special feature that lattice two-scale difference equations make sense when restricted to the discrete domain $\mathbb{Z}$. Suppose we are given data $\{f(n);\,n\in\mathbb{Z}\}$ which satisfy

$$f(x)=\sum_{n=0}^{N} c_n f(kx-n) \tag{1.6}$$

for all $x\in\mathbb{Z}$. The functional equation then determines $f(x)$ for $x\in\mathbb{Z}/k$, and by iteration for $x\in\bigcup_{n=1}^{\infty}\mathbb{Z}/k^n$. In particular, such data $\{f(n);\,n\in\mathbb{Z}\}$ can be interpolated by at most one continuous solution of (1.6). This approach thus applies most naturally to the problem of finding *continuous* solutions of (1.6). Here we have the two subproblems of finding solutions to (1.6) on $\mathbb{Z}$, and then determining conditions under which such solutions interpolate to solutions on $\mathbb{R}$.

The time-domain approach applies particularly well in the study of compactly supported continuous solutions, since we then know that $\{f(n);\, n\in\mathbb{Z}\}$ has $f(n)=0$ off the finite set $0\leqq n\leqq N/(k-1)$, and the set of solutions on $\mathbb{Z}$ to (1.6) is a finite-dimensional vector space. The iterative process of recursively solving (1.6) on $\{\mathbb{Z}/k^n;\, n=1,2,\cdots\}$ can be encoded using products of a finite set of matrices, as is explained in part II, and this provides a vehicle for studying convergence and smoothness of solutions. Such an approach was initiated by Micchelli and Prautzsch (1987a), as we discovered after completing this work.

In the rest of part I we apply the time-domain approach to obtain information about compactly supported solutions in the case where $\Delta=1$, which by the results of § 3 is essentially the most general case.

Section 4 obtains results on two different iterative methods to find solutions of the lattice two-scale equation (1.6). A solution is a fixed point $f=Vf$ of the linear operator

$$Vf(x)=\sum_{n=0}^{N} c_n f(kx-n), \tag{1.7}$$

and a natural approach is to consider iterative schemes $f_j=Vf_{j-1}$ that converge to a fixed point f starting from suitable f_0. Given data $\{f(n);\, n\in\mathbb{Z}\}$ for an L^1-solution of such an equation with $\Delta=1$, we can construct a piecewise linear spline f_0 that has $f_0(n)=f(n)$ for all $n\in\mathbb{Z}$. We show that if f is continuous, then the iterates $f_{j+1}=Vf_j$ are piecewise linear splines with successively finer knot sets (Theorem 4.1) and that $f_j\to f$ pointwise, with a rate of convergence depending on the smoothness of f. If f is L times continuously differentiable, then f_0 can be chosen to be a C^L piecewise polynomial spline of degree $2L+1$, with $f_0^{(l)}(n)=f^{(l)}(n)$ for all $n\in\mathbb{Z}$, $l=0,\cdots,L$. Then the iterates $f_{j+1}=Vf_j$ are again C^L piecewise polynomial splines of degree $2L+1$ with successively finer knot sets, and we prove $f_j^{(l)}\to f^{(l)}$ pointwise, for all $l=0,\cdots,L$. These results show in particular that convergence to a C^0-solution $f(x)$ occurs when one exists, if we start with correct initial conditions on $\mathbb{Z}$. However, they give no information concerning existence of such solutions.

The second iterative method for finding solutions to (1.6) discussed in § 4 is the "cascade algorithm." The successive approximations f_j in this scheme are again defined by $f_j=Vf_{j-1}$, but the starting point is now $f_0(x)=1-|x|$ for $-\frac{1}{2}\leqq x\leqq\frac{1}{2}$ and $f_0(x)=0$ otherwise. These initial conditions are not usually a solution to (1.6) on $\mathbb{Z}$. The advantage of the cascade algorithm is that f_j can be computed via a "local" method: at every step j, the value of $f_j(x)$ can be determined using *only* the values obtained in the previous step in the region $\{y;\, |y-x|\leqq C2^{-j}\}$ (C independent of j), a neighborhood of x becoming exponentially small as j increases. This lends a "zoom-in" quality to the successive steps of the cascade algorithm (when it converges). It is known that the cascade algorithm does not always converge pointwise to a nonzero C^0-solution when one exists. This scheme has been studied by several authors (cf. Deslauriers and Dubuc (1989), Dyn, Gregory, and Levin (1989), (1990)), and various sufficient conditions for its convergence are known.

Section 5 obtains bounds on the global regularity of any nonzero L^1-solution to (1.6) when $\Delta=1$. Theorem 5.1 shows that if such a solution is in $C^m(\mathbb{R})$, then $m<N/(k-1)-1$. This result is best possible in the sense that there exist equations having C^m-solutions, for which $m\geqq N/(k-1)-2$, for arbitrarily large m.

Finally, § 6 applies these results to three examples.

A number of authors have studied more general solutions to two-scale difference equations and related functional equations. In constructing fractals Barnsley and Demko (1985) study measures μ which are solutions of

$$\mu(S)=\sum_{n=0}^{N} c_n\mu(\alpha S-\beta_n), \qquad S \text{ a Borel set.}$$

The *adjoint operator* (in the L^2-sense) to (1.1) is

$$V^A g(x)=\sum_{n=0}^{N} c_n g(\alpha^{-1}(x+\beta_n)).$$

The stationary measures μ studied by Diaconis and Shashahani (1986) are fixed points of generalizations of such adjoint operators. It is also interesting to note that the mth Bernoulli polynomial $B_m(x)$ satisfies the equation $V^A B_m(x)=B_m(x)$ with

$$V^A g(x)=\sum_{n=0}^{m-1} n^{m-1} g\left(\frac{1}{m}(x+n)\right).$$

2. Existence and uniqueness of L^1-solutions. We are interested in L^1-solutions f to the two-scale difference equation

$$f(x)=\sum_{n=0}^{N} c_n f(\alpha x-\beta_n). \tag{2.1}$$

Since f is in $L^1(\mathbb{R})$, its Fourier transform $\hat{f}$,

$$\hat{f}(u)=\int_{-\infty}^{\infty} e^{ixu} f(x)\, dx,$$

is a bounded continuous function. By viewing (2.1) as a convolution equation, we see that $\hat{f}$ satisfies

$$\hat{f}(u)=P(\alpha^{-1}u)\hat{f}(\alpha^{-1}u) \tag{2.2}$$

where

$$P(u)=\frac{1}{\alpha}\sum_{n=0}^{N} c_n\, e^{i\beta_n u}. \tag{2.3}$$

The existence and uniqueness of L^1-solutions to (2.1) are governed by the value

$$\Delta=P(0)=\frac{1}{\alpha}\sum_{n=0}^{N} c_n,$$

as shown in the following result.

THEOREM 2.1. *Let Δ be defined as above. Then the following are true.*

(a) *If $|\Delta|\leqq 1$ and $\Delta\neq 1$, then the only L^1-solution of* (2.1) *is the trivial solution $f\equiv 0$.*

(b) *If $\Delta=1$ then there exists, up to normalization, at most one nontrivial L^1-solution to* (2.1). *If it exists, then its Fourier transform is given by*

$$\hat{f}(u)=A\prod_{j=1}^{\infty} P(\alpha^{-j}u) \tag{2.4}$$

where $A=\hat{f}(0)=\int f(x)\,dx$ and the infinite product converges for all u. Conversely, if the right-hand side of (2.4) *is the inverse Fourier transform of an L^1-function f, then f is a nontrivial L^1-solution to* (2.1).

(c) *If* $|\Delta|>1$, *then the Fourier transform of any* L^1*-solution* f *is necessarily of the form*

$$\hat{f}(u)=\left[\prod_{j=1}^{\infty} p(\alpha^{-j}u)\right]\exp\left(\frac{\ln \Delta}{\ln \alpha}\ln|u|\right) g_{\operatorname{sgn}(u)}\left(\frac{\ln|u|}{\ln \alpha}\right) \tag{2.5}$$

where

$p(u)=\Delta^{-1}P(u)$,

$g_{\pm}$ *are continuous periodic functions with period* 1, *and*

$\ln \Delta=\ln|\Delta|+i\theta$ *where* $\Delta=|\Delta|\, e^{i\theta}$ *and* $-\pi<\theta\leqq\pi$.

Furthermore, the infinite product converges for all complex u. *Conversely, if* g_+, g_- *are continuous periodic functions of period* 1 *such that the inverse Fourier transform* f *of the right side of* (2.5) *is in* $L^1(\mathbb{R})$, *then* f *satisfies* (2.1).

Proof. (1) We have

$$\begin{aligned}|P(u)-\Delta| &\leqq \alpha^{-1}\sum_{n=0}^{N}|c_n||e^{i\beta_n u}-1| \\ &\leqq K\min(1,|u|)\exp[B|\operatorname{Im}(u)|]\end{aligned} \tag{2.6}$$

where $B=\max|\beta_i|$ and $K=2\alpha^{-1}\sum_{n=0}^{N}|c_n|(1+|\beta_n|)$.

(2) We first treat the case where $|\Delta|<1$. Since $f\in L^1(\mathbb{R})$, $\hat{f}(u)$ is continuous for $u\in\mathbb{R}$. From (2.6) we have (for real u)

$$|\hat{f}(u)|=|P(\alpha^{-1}u)||\hat{f}(\alpha^{-1}u)|\leqq(|\Delta|+K\alpha^{-1}|u|)|\hat{f}(\alpha^{-1}u)|.$$

It follows that, for all $j\in\mathbb{N}$,

$$|\hat{f}(u)|\leqq\|f\|_1\prod_{l=1}^{j}(|\Delta|+K\alpha^{-l}|u|). \tag{2.7}$$

For any real u we can make the product on the right side of (2.7) arbitrarily small by choosing j large enough, since $|\Delta|<1$. Hence $f\equiv 0$. This proves (a), except for $|\Delta|=1$, which we treat below.

(3) For $|\Delta|\geqq 1$ define $p(u)=\Delta^{-1}P(u)$. By (2.6) we have

$$|p(u)-1|\leqq K\, e^{B}\Delta^{-1}|u|:=K'|u|, \tag{2.8}$$

for complex $|u|<1$. Now we define

$$\hat{f}_0(u):=\prod_{j=1}^{\infty}p(\alpha^{-j}u), \tag{2.9}$$

and (2.8) shows that the infinite product converges absolutely and uniformly on compact subsets of $\mathbb{C}$ to an entire function. The bound (2.8) shows that for $|u|\leqq(2K')^{-1}$, $|p(u)|\geqq 1-K'|u|\geqq(1+2K'|u|)^{-1}$ and

$$\begin{aligned}|\hat{f}(\alpha^{-j}u)| &= |\hat{f}(u)|\prod_{l=1}^{j}|P(\alpha^{-l}u)|^{-1} \\ &\leqq |\Delta|^{-j}|\hat{f}(u)|\prod_{l=1}^{j}(1+2K'\alpha^{-l}|u|) \\ &\leqq |\Delta|^{-j}\|f\|_1\exp[2K'(\alpha-1)^{-1}|u|].\end{aligned}$$

This implies that

$$|\hat{f}(u)|\leqq\exp[(\alpha-1)^{-1}]\|f\|_1|\Delta|^{-j}\quad\text{when } |u|\leqq(2K')^{-1}\alpha^{-j}.$$

For any $|u| \leqq (2K')^{-1}$ we can find j so that $(2K')^{-1}\alpha^{-(j+1)} \leqq |u| \leqq (2K')^{-1}\alpha^{-j}$. It follows that

$$\begin{aligned} |\hat{f}(u)| &\leqq \exp[(\alpha-1)^{-1}]\|f\|_1 |\Delta|^{-j} \\ &\leqq \exp[(\alpha-1)^{-1}]\|f\|_1 \alpha^{-j\ln|\Delta|/\ln\alpha} \\ &\leqq C|u|^{\ln|\Delta|/\ln\alpha} \end{aligned} \tag{2.10}$$

for all $|u| \leqq (2K')^{-1}$. Since $|\hat{f}(u)|$ is bounded for real u ($f \in L^1$), (2.10) also extends (with possibly a different constant) to all $u \in \mathbb{R}$. Note that for $|\Delta| > 1$, (2.10) implies that

$$\int_{-\infty}^{\infty} f(x)\,dx = \hat{f}(0) = 0,$$

which can also be obtained directly from (2.1) by integration. Define $F(u)$ by

$$F(u) := \exp\left(-\frac{\ln|\Delta|}{\ln\alpha}\ln|u|\right)\hat{f}(u).$$

Since $\hat{f}$ is continuous, F is continuous as well, except possibly at $u=0$, and by (2.10), F is bounded near $u=0$. The function F satisfies the recursion

$$F(u) = e^{i\theta}p(\alpha^{-1}u)F(\alpha^{-1}u), \tag{2.11}$$

where $\Delta = |\Delta|\, e^{i\theta}$ and $-\pi < \theta \leqq \pi$. This yields

$$F(u) = \left[\prod_{j=1}^{J} p(\alpha^{-j}u)\right] e^{iJ\theta}F(\alpha^{-J}u).$$

The first factor has the limit $\hat{f}_0(u)$ as $J \to \infty$. When this limit is not zero, the second factor must also converge as $J \to \infty$, and we denote this limit by $\phi(u)$, so that

$$\phi(u) := [\hat{f}_0(u)]^{-1}F(u).$$

The function ϕ is continuous, except possibly at $u=0$, where F may be discontinuous, and at the zeros of $\hat{f}_0(u)$. From (2.8) we find for complex $|u| \leqq 1$ that

$$|\hat{f}_0(u) - 1| \leqq K'|u| \exp[B(\alpha-1)^{-1}|u|],$$

which implies that $\hat{f}_0(u)$ is bounded away from zero in a neighborhood of $u=0$. It follows that ϕ is continuous in a neighborhood of zero, except possibly at $u=0$. On the other hand, the recursion (2.11) for F gives

$$\phi(u) = e^{i\theta}\phi(\alpha^{-1}u).$$

Define the two functions

$$g_{\pm}(t) := \phi(\pm\alpha^{t})\exp(i\theta t), \qquad t \in \mathbb{R}.$$

Then

$$g_{\pm}(t+1) = g_{\pm}(t). \tag{2.12}$$

If the periodic functions $g_{\pm}$ had any singularity (including discontinuities), then ϕ would have infinitely many singularities in a neighborhood of $u=0$. Since this is impossible, it follows that the $g_{\pm}$ are continuous. To prove the converse statement, it suffices to observe that, under the stated conditions, for $|\Delta| \geqq 1$, the right-hand side of (2.5) satisfies the functional equation $\hat{f}(u) = P(\alpha^{-1}u)\hat{f}(\alpha^{-1}u)$. This establishes (c).

(4) If $|\Delta| = 1$, then the above construction simplifies. We have

$$\hat{f}(u) = \hat{f}_0(u)\phi(u), \tag{2.13}$$

where ϕ satisfies

$$\phi(u)=\Delta\phi(\alpha^{-1}u)=e^{i\theta}\phi(\alpha^{-1}u).$$

Since $\hat{f}_0(u)$ is bounded away from zero for small $|u|$, it follows from (2.13) that ϕ is continuous at zero. In particular,

$$\phi(0)=e^{i\theta}\phi(0),$$

implying either $\Delta=e^{i\theta}=1$ or $\phi(0)=0$. However, we also know that

$$\phi(u)=g_{\operatorname{sgn}(u)}\left(\frac{\ln|u|}{\ln\alpha}\right)\exp\left(i\theta\frac{\ln|u|}{\ln\alpha}\right)$$

where $g_\pm$ are continuous periodic functions with period 1. If $\Delta\neq 1$ then $\phi(0)=0$; hence $|g_\pm(t)|=|\phi(\pm\alpha^t)|\to 0$ as $t\to-\infty$. Since the functions $g_\pm$ are periodic, this forces $g_\pm\equiv 0$; hence $\phi\equiv 0$ and $f\equiv 0$ for $\Delta\neq 1$, which proves the rest of (a). If $\Delta=1$ then $g_\pm(t)=\phi(\pm\alpha^t)\to\phi(0)$ as $t\to\infty$. By the periodicity of $g_\pm$ this implies that $g_\pm\equiv\phi(0)$ are both the same constant function; hence $\hat{f}(u)=\phi(0)\hat{f}_0(u)$. This proves (b). □

Remarks. (1) Theorem 2.1 also holds for "infinite" two-scale difference equations, i.e.,

$$f(x)=\sum_{n=-\infty}^{\infty}c_nf(\alpha x-\beta_n)$$

provided that $\sum_{n=-\infty}^{\infty}|c_n||\beta_n|^\delta<\infty$ for some $\delta>0$. The estimate (2.6) then becomes

$$|P(u)-\Delta|\leqq\alpha^{-1}\sum_{n=-\infty}^{\infty}|c_n|\min(2,|\beta_nu|^{\min(1,\delta)})\,e^{|B||\operatorname{Im}u|},$$

and the other estimates can be adjusted similarly.

(2) Note that there may exist distributional solutions even if $|\Delta|<1$. One example is the equation

$$f(x)=\tfrac{1}{12}f(2x)+\tfrac{1}{6}[f(2x+1)+f(2x-1)]-\tfrac{1}{12}[f(2x+2)+f(2x-2)],$$

which admits $f(x)=x^2$ as a solution. The Fourier transform of this solution is a distribution supported at the origin, so that the continuity argument used in the proof of Theorem 2.1 does not apply.

(3) There are no distributional solutions with Fourier transform continuous at zero if $|\Delta|<1$. For $|\Delta|\geqq 1$, (2.4) and (2.5) always give distributional solutions to the two-scale equation (2.1), but there may exist other distributional solutions with discontinuous Fourier transforms.

Theorem 2.1 has the following corollary, proved in the lattice case by Deslauriers and Dubuc (1987).

COROLLARY 2.2. *If the two-scale difference equation* (2.1) *with* $\Delta=1$ *possesses a nontrivial* L^1*-solution* f, *then* $\int_{-\infty}^{\infty}f(x)\,dx\neq 0$ *and* f *has compact support, with*

$$\operatorname{supp}(f)\subset[\beta_0(\alpha-1)^{-1},\beta_N(\alpha-1)^{-1}]. \tag{2.14}$$

Proof. By Theorem 2.1

$$\hat{f}(u)=\hat{f}(0)\prod_{j=1}^{\infty}P(\alpha^{-j}u)=\hat{f}(0)\hat{f}_0(u).$$

Hence $\int_{-\infty}^{\infty} f(x)\,dx = \hat{f}(0) \neq 0$. We can without loss of generality reduce to the case that $\beta_0 = -\beta_N$ by considering

$$f_1(x) = f\left(x - (\alpha-1)^{-1}\frac{\beta_N - \beta_0}{2}\right),$$

which is easily checked to satisfy

$$f_1(x) = \sum_{n=0}^{N} c_n f_1\left(\alpha x - \beta_j + \frac{\beta_0 - \beta_N}{2}\right).$$

Now suppose $\beta_0 = -\beta_N$ and (2.6) becomes

$$|P(u) - 1| \leqq K \min(1, |u|) \exp(B|\mathrm{Im}(u)|)$$

with $B = \beta_N$. On the annulus $\alpha^k \leqq |u| \leqq \alpha^{k+1}$ this gives

$$\begin{aligned} |\hat{f}_0(u)| &\leqq \prod_{j=1}^{k+1} [1 + K\Delta^{-1} \exp(B\alpha^{-j}|\mathrm{Im}(u)|)] \prod_{j=k+2}^{\infty} [1 + K'\alpha^{-j} \exp(B)] \\ &\leqq C(1+|u|)^M \exp[B(\alpha-1)^{-1}|\mathrm{Im}(u)|], \end{aligned} \tag{2.15}$$

where $M = [|\ln((K\Delta^{-1}+1)/\alpha)| + 1]$ and C is a constant. By the Paley-Wiener theorem for distributions (see, e.g., Reed and Simon (1975, Thm. IX.12)) $\hat{f}_0(u)$ is the Fourier transform of a distribution f_0 in $\mathcal{S}'(\mathbb{R})$ having compact support in the interval $|x| \leqq B(\alpha-1)^{-1}$. By hypothesis this distribution is the L^1-function $[\hat{f}(0)]^{-1}f$; hence f has compact support in $|x| \leqq (\beta_n - \beta_0)/(2(\alpha-1))$ and (2.14) follows. □

This proof shows that *all* two-scale difference equations with $\Delta = 1$ possess a distributional solution f in $\mathcal{S}'(\mathbb{R})$ having compact support in $[\beta_0(\alpha-1)^{-1}, \beta_N(\alpha-1)^{-1}]$ which has Fourier transform (2.4). The arguments of § 3 will show that up to a scale factor this is the unique distribution in $\mathcal{S}'(\mathbb{R})$ which satisfies (3.1) and has compact support. The following examples illustrate a few cases, for different values of Δ.

Examples. (1) Consider the lattice two-scale difference equation

$$f(x) = \tfrac{1}{2}\Delta[f(2x) + 2f(2x-1) + f(2x-2)].$$

Depending on the value of Δ, there will be one, infinitely many, or no nontrivial L^1-solutions. If $\Delta = 1$, then any candidate L^1-solution satisfies

$$\begin{aligned} \hat{f}(u) &= \hat{f}(0) \prod_{j=1}^{\infty} \left\{\frac{1}{4}[1 + \exp(i2^{-j}u)]^2\right\} \\ &= \hat{f}(0)\, e^{iu} \left(\frac{\sin(u/2)}{u/2}\right)^2. \end{aligned}$$

It follows that f is a multiple of the function g,

$$g(x) = \begin{cases} x, & 0 \leqq x \leqq 1, \\ 2-x, & 1 \leqq x \leqq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Up to normalization, we therefore have a unique L^1-solution in this case. For $\Delta = 2$, we find

$$\hat{f}(u) = |u|\, e^{iu} \left(\frac{\sin(u/2)}{u/2}\right)^2 g_{\mathrm{sgn}(u)}\left(\frac{\ln|u|}{\ln 2}\right),$$

where $g_\pm$ are periodic, continuous functions of period 1. Clearly, $\hat{f} \in L^2(\mathbb{R})$. If $g_\pm$ are C^1, then we easily check that also $(\hat{f})' \in L^2$, which implies $f \in L^1$. There is therefore clearly an infinity of different possible L^1-solutions in this case. Only one of these L^1-solutions is compactly supported (see § 3). For $\Delta = 4$, however, we have

$$\hat{f}(u) = 4\, e^{iu}[\sin(u/2)]^2 g_{\operatorname{sgn}(u)}\left(\frac{\ln|u|}{\ln 2}\right).$$

This tends to zero for $|u| \to \infty$ only if both functions $g_\pm \equiv 0$. The only L^1-solution is therefore the trivial solution $f \equiv 0$.

(2) The following example shows that $\Delta = 1$ does not imply the existence of a nontrivial L^1-solution. Take the lattice two-scale difference equation

$$f(x) = 2f(2x-1).$$

Every candidate L^1-solution satisfies

$$\hat{f}(u) = \hat{f}(0) \prod_{j=1}^{\infty} [\exp(i2^{-j}u)] = \hat{f}(0)\, e^{iu}.$$

Since e^{iu} is the Fourier transform of the Dirac δ-measure at $x = 1$, there are no nontrivial L^1-solutions.

(3) Consider the family of two-scale difference equations

$$f_\alpha(x) = \frac{\alpha}{2}\{f_\alpha(\alpha x - 1) + f_\alpha(\alpha x + 1)\},$$

which all have $\Delta = 1$. This equation always has a distributional solution with Fourier transform

$$\hat{f}_\alpha(u) = \prod_{n=1}^{\infty} \cos(\alpha^{-n} u),$$

which has compact support in $[-(\alpha-1)^{-1}, (\alpha-1)^{-1}]$ by the same argument as in the proof of Corollary 2.2. The smoothness of this distribution as a function of $\beta = \alpha^{-1}$ for $0 < \beta < 1$ was studied by Kershner and Wintner (1935), Erdös (1939), (1940), and Garsia (1962). It is known that for $\alpha = 2^{1/k}$, with k sufficiently large, the function f_α is continuous (hence in $L^1(\mathbb{R})$) and arbitrarily smooth. Erdös (1940) showed that for any k there is a constant $c(k)$ such that for almost all $\beta = \alpha^{-1}$ in the interval $[c(k), 1]$ the distribution f_α is a function in $C^{(k)}(\mathbb{R})$.

3. L^1-solutions having compact support. We consider the general two-scale difference equation

$$f(x) = \sum_{n=0}^{N} c_n f(\alpha x - \beta_n), \tag{3.1}$$

and derive necessary conditions for the existence of nonzero L^1-function of compact support.

THEOREM 3.1. *Suppose that the two-scale difference equation* (3.1) *possesses a nonzero L^1-solution f having compact support. Then*:

(a) *$\Delta = \alpha^m$ for a nonnegative integer m.*

(b) *f is unique up to a scale factor and has Fourier transform*

$$\hat{f}(u) = A u^m \prod_{k=1}^{\infty} p(\alpha^{-k} u) \tag{3.2}$$

where $p(u) = \Delta^{-1} \sum_{n=0}^{n} c_n\, e^{i\beta_n u}$.

(c) *The two-scale equation with* $\Delta = 1$ *obtained by replacing* $\{c_n\}$ *by* $\{\alpha^{-m}c_n\}$ *has a nonzero* L^1*-solution* g *unique up to scale, and with a proper choice of scale* $(d^m/dx^m)g(x) \equiv f(x)$.

The main ingredient in the proof is the following result.

LEMMA 3.2. *Suppose that the function* M_0, *defined on* $\mathbb{R}-\{0\}$ *by*

$$M_0(u) := \exp(\gamma_0 \ln|u|) g_{\mathrm{sgn}(u)}(\gamma_1 \ln|u|), \tag{3.3}$$

is such that

(1) $g_\pm(t)$ *are periodic functions of period* 1.

(2) γ_1 *is real.*

(3) $M_0(u)$ *possesses an analytic continuation to* $\mathbb{C}$ *which is an entire function of exponential type.*

Then $M_0(u) = Au^m$ *where* m *is a nonnegative integer.*

Proof. (1) The function $h(u) = \exp(\gamma_0 \ln u)$ can be continued analytically to the simply-connected two-sheeted region $\mathbf{R} = \{z = r\,e^{i\theta};\ r > 0 \text{ and } -\frac{3}{2}\pi \leqq \theta \leqq \frac{3}{2}\pi\}$ where $r\,e^{i\theta}$ and $r\,e^{i(\theta+2\pi)}$ are viewed as distinct points and

$$h(r\,e^{i(\theta+2\pi)}) = \exp(2\gamma_0\pi i) h(r\,e^{i\theta}), \tag{3.4}$$

whenever both sides are in $\mathbf{R}$.

(2) Since $M_0(u)$ is entire and $h(z)$ is nonzero on $\mathbf{R}$, it follows from (3.3) that $g_+(\gamma_1 \ln u)$ has an analytic continuation to $\mathbf{R}$. Therefore, in terms of the variable $t = \gamma_1 \ln u$, $g_+(t)$ has an analytic continuation to the horizontal strip $\mathbf{T} = \{t;\ -\frac{3}{2}\pi\gamma_1 < \mathrm{Im}(t) < \frac{3}{2}\pi\gamma_1\}$. The periodicity

$$g_+(t+1) = g_+(t) \tag{3.5}$$

on the real axis extends to this strip by analytic continuation. Now the single-valuedness of $M_0(u)$ on $\mathbb{C}$ means that

$$M_0(r\,e^{i(\theta+2\pi)}) = M_0(r\,e^{i\theta}), \qquad i = 1, 2,$$

on the region $\mathbf{R}$, and combined with (3.3) and (3.4) this implies that

$$g_+(t + 2\gamma_1\pi i) = \exp(-2\gamma_0\pi i) g_+(t) \tag{3.6}$$

is valid when both t, $t + 2\gamma_1\pi i \in \mathbf{T}$, i.e., for $\{t;\ -\frac{3}{2}\pi\gamma_1 < \mathrm{Im}(t) < \frac{1}{2}\pi\gamma_1\}$. This relation allows us to continue g_+ analytically to the entire plane, and (3.5), (3.6) then hold for all complex t.

(3) We claim next that $g_+(t)$ is an entire function of exponential type. To see this, note that it is bounded by a constant, say C, on the rectangle $0 \leqq \mathrm{Re}(t) \leqq 1$, $0 \leqq \mathrm{Im}(t) \leqq 2\pi\gamma_1$, and the periodicity relations (3.5) and (3.6) then give

$$|g_+(t)| \leqq C \exp(2\pi|\gamma_0||\mathrm{Im}(t)|),$$

proving the claim.

Next we show that $g_+(t)$ has no zeros. For if it had a zero at t_0, the periodicity relations (3.5) and (3.6) would give it zeros at $t_0 + m + (2\gamma_1\pi i)n$ for m, $n \in \mathbb{Z}$, which contradicts the property that an entire function of exponential type has $O(R)$ zeros in the disc $\{t: |t| \leqq R\}$ as $R \to \infty$.

Since $g_+(t)$ is an entire function of exponential type having no zeros, $g_+(t) = A_+ \exp(c_+ z)$ for some constant c_+. The periodicity (3.5) forces $c_+ = 2k\pi i$ for some $k \in \mathbb{Z}$.

(4) A similar argument applied to $g_-(\gamma_1 \ln(-u))$ shows that $g_-(t) = A_- \exp(c_- z)$ where $c_- = 2l\pi i$ for some $l \in \mathbb{Z}$.

(5) Now we have for real u that

$$M_0(u) = \begin{cases} A_+ \exp[(\gamma_0 + 2k\pi i) \ln u], & u > 0, \\ A_- \exp[(\gamma_0 + 2l\pi i) \ln(-u)], & u < 0. \end{cases}$$

The first expression has a singularity at $u = 0$ unless $\gamma_0 + 2k\pi i = m_+$ is an integer. Similarly, we conclude that $\gamma_0 + 2l\pi i = m_-$ is an integer. Analyticity of $M_0(u)$ at $u = 0$ yields $A_+ = A_-$ and $m_+ = m_- = m \geqq 0$, and $M_0(u) = Au^m$ for a nonnegative integer m. □

We now proceed to prove Theorem 3.1.

Proof of Theorem 3.1. (1) By the Paley–Wiener theorem the Fourier–Laplace transform $\hat{f}$ of f is necessarily an entire function of exponential type, satisfying

$$|\hat{f}(u)| \leqq C \exp(B|\mathrm{Im}\, u|) \tag{3.7}$$

for some constants B, C. Now set

$$\hat{f}_0(u) := \prod_{j=1}^{\infty} p(\alpha^{-j} u),$$

which by the argument in the proof of Corollary 2.2 is also entirely of exponential type. We claim that

$$M(u) := \frac{\hat{f}(u)}{\hat{f}_0(u)} \tag{3.8}$$

is also an entire function of exponential type, i.e., any zero of $\hat{f}(u)$ has at least the multiplicity of $\hat{f}_0(u)$. To see this, note that for any zero u_0 of $\hat{f}_0(u)$ of multiplicity m there is a finite product $\prod_{j=1}^{J} p(\alpha^{-j}u)$ having a zero of the same multiplicity. Iterating the basic recursion (2.2) yields

$$\hat{f}(u) = \Delta^J \left(\prod_{j=1}^{J} p(\alpha^{-j} u) \right) \hat{f}(\alpha^{-J} u).$$

Since all terms on the right side of this expression are analytic at u_0 and the product has a zero of multiplicity m, $\hat{f}(u)$ has a zero there of at least that multiplicity, and the claim follows.

(2) Since $f \in L^1$, it satisfies the formula of Theorem 2.1(c). Thus the hypotheses of Lemma 3.2 are satisfied for $M(u)$ given by (3.8). Consequently, $M = Au^m$ and

$$\hat{f}(u) = Au^m \prod_{k=1}^{\infty} p(\alpha^{-k} u). \tag{3.9}$$

This proves claim (b). On the other hand, the two-scale equation (3.1) implies

$$\hat{f}(u) = \Delta p(\alpha^{-1} u) \hat{f}(\alpha^{-1} u).$$

Substituting (3.9) gives $\Delta = \alpha^m$, which proves (a).

(3) For (c), observe that if $m \geqq 1$ then $\hat{f}(0) = 0$; hence

$$\int_{-\infty}^{\infty} f(x)\, dx = 0. \tag{3.10}$$

Define $f_1(x) = \int_{-\infty}^{x} f(w)\, dw$ and observe that since f has compact support (3.10) shows that $f_1(x)$ is in $L^1(\mathbb{R})$ with compact support. Furthermore, f_1 satisfies the two-scale equation (3.1) with $\{c_n\}$ replaced by $\{\alpha^{-1} c_n\}$, by integrating (3.1). Of course $(d/dx)(f_1(x)) = f(x)$. By integrating m times, (c) follows. □

Remarks. (1) Under the weaker hypothesis that (3.1) possesses a solution which is a tempered distribution of compact support, the conclusions (a) and (b) of the theorem still hold.

(2) L^1-solutions satisfying (a) can exist for arbitrarily large values of m, for suitable values of α and of the c_j, β_j. By (c), this is equivalent to saying that there exist L^1-solutions with arbitrary high regularity for two-scale difference equations with $\Delta = 1$. Examples are given in § 6.

4. Lattice two-scale difference equations: iterative approximations. In the remainder of this paper we study compactly supported continuous solutions of lattice two-scale difference equations,

$$f(x) = \sum_{n=0}^{N} c_n f(kx-n), \tag{4.1}$$

where k is an integer $\geqq 2$. We will suppose that $\Delta = (1/k) \sum_{n=0}^{N} c_n = 1$, which involves essentially no loss of generality by Theorem 3.1.

A continuous solution of such an equation is a fixed point $Vf = f$ of the linear operator

$$Vf(x) = \sum_{n=0}^{N} c_n f(kx-n) \tag{4.2}$$

acting on a function space, e.g., $C^0(\mathbb{R})$. A natural method to construct a solution of (4.1) is as a limit of the iterative approximation scheme $f_{j+1} = Vf_j$, where f_0 is a suitable initial function. In this section we discuss the convergence of two such approximation schemes.

We first suppose that a compactly supported continuous solution $f(x)$ exists, and that the data $\{f(n): n \in \mathbb{Z}\}$ are known. We consider initial functions f_0 which are piecewise linear splines interpolating these data with knots at the integers $\mathbb{Z}$. That is, f_0 is defined by

$$f_0(x) = f(n)(n+1-x) + f(n+1)(x-n) \quad \text{for } n \leqq x \leqq n+1.$$

Since $f(n) = 0$ for $n \notin [0, (k-1)^{-1}N]$, f_0 has compact support in $[0, \lceil (k-1)^{-1}N \rceil]$, so we may regard it as being defined on the finite knot set $\mathbb{Z} \cap [0, \lceil (k-1)^{-1}N \rceil]$. (As usual $\lceil a \rceil$ stands for "smallest integer larger than or equal to a.") It immediately follows that $f_j = V^j f_0$ is a piecewise linear spline with knots at the $k^{-j}n$, $0 \leqq n \leqq \lceil k^j (k-1)^{-1} N \rceil$, which agrees with f at these knots. Consequently we have Theorem 4.1.

THEOREM 4.1. *Suppose that the lattice two-scale equation* (4.1) *with* $\Delta = 1$ *has a nonzero continuous solution f of compact support. Let f_0 be the spline of degree* 1 *with knot set* $\{n; n \in \mathbb{Z} \cap [0, \lceil (k-1)^{-1} N \rceil]$, *and with* $f_0(n) = f(n)$. *Define* $f_j = V^j f_0$, *with V as in* (4.2). *Then*

(1) *f_j is an interpolating spline of degree* 1 *with knot set* $k^{-j}(\mathbb{Z} \cap [0, \lceil k^j (k-1)^{-1} N \rceil])$.

(2) *f_j agrees with f at its knots,* $f_j(k^{-j}n) = f(k^{-j}n)$.

(3) $\|f - f_j\|_{L^\infty} \to 0$ *as* $j \to \infty$.

(4) *If* $f \in \mathrm{Lip}^\alpha$ *for* $0 < \alpha \leqq 1$, *i.e.,* $|f(x) - f(y)| \leqq C|x-y|^\alpha$, *then* $\|f - f_j\|_{L^\infty} \leqq Ck^{-j\alpha}$.

Proof. (1) and (2) were derived above; (3) and (4) are standard spline convergence results; see, e.g., Schumaker (1981, Thm. 6.15) or Theorem 4.2 below. □

If the compactly supported solution f has more regularity, e.g., if $f \in \mathrm{Lip}^{L,\alpha}$ (which means $f \in C^L$ and $d^L f/dx^L \in \mathrm{Lip}^\alpha$), then the same piecewise linear f_j converge even faster to f (Schumaker (1981, Thm. 6.15)). In order to obtain convergence of the derivatives as well, we need to use an initial function f_0 that is more regular. This can be achieved by choosing for f_0 a C^L piecewise polynomial spline of degree $2L+1$ that agrees with f and its first L derivatives on the knot set $\mathbb{Z}$. (Similar fast convergence of the f_j and their derivatives can be achieved by other C^L choices for f_0, for which the derivatives on $\mathbb{Z}$ do not necessarily agree with those of f. In our present case, however, we can determine the $f^{(l)}(n)$ easily, and we can therefore afford to pick this particular f_0. The associated f_j will play a role in part II as well.)

THEOREM 4.2. *Suppose that the lattice two-scale equation* (4.1) *with* $\Delta=1$ *has a nonzero solution f of compact support which is L times continuously differentiable. Let* f_0 *be the* C^L *interpolating spline of degree* $2L+1$ *with knot set* $\{n;\, n\in\mathbb{Z}\cap[0, \lceil (k-1)^{-1}N\rceil]\}$ *and such that* $f_0^{(l)}(n)=f^{(l)}(n)$, $l=0,\cdots,L$. *Define* $f_j=V^jf_0$, *with V as in* (4.2). *Then*

(1) f_j *is a* C^L *interpolating spline of degree* $2L+1$ *with knot set* $k^{-j}(\mathbb{Z}\cap[0, \lceil k^j(k-1)^{-1}N\rceil])$.

(2) $f_j^{(l)}(k^{-j}n)=f^{(l)}(k^{-j}n)$ *for* $n\in\mathbb{Z}$ *and,* $l=0,\cdots,L$.

(3) *For all l,* $0\leqq l\leqq L$, $\|f^{(l)}-f_j^{(l)}\|_{L^\infty}\leqq Ck^{-j(L-l)}$.

(4) *If* $f\in \mathrm{Lip}^{L,\alpha}$, *then* $\|f^{(l)}-f_j^{(l)}\|_{L^\infty}\leqq Ck^{-j(L-l+\alpha)}$ *for* $l=0,\cdots,L$.

Proof. Note first that f_0 exists and is uniquely determined by the constraints imposed: on every interval $[n, n+1]$, the $2L+2$ coefficients of f_0 are linear functions of the $2L+2$ boundary values $f_0^{(l)}(n), f_0^{(l)}(n+1)$, $l=0,\cdots,L$. It is obvious that $f_0\in C^L$. It then immediately follows that f_j is also C^L, that f_j is piecewise polynomial of degree $2L+1$, with knots at $k^{-j}\mathbb{Z}$, and that $f_j^{(l)}(k^{-j}n)=f^{(l)}(k^{-j}n)$, for $l=0,\cdots,L$. Points 3 and 4 are again standard results in spline approximation theory (they can, e.g., easily be proved by methods similar to those used in the proof of Theorem 6.15 in Schumaker (1981)); for the sake of convenience we also give an explicit and simple proof in the Appendix. □

Theorems 4.1 and 4.2 guarantee convergence of spline interpolants, provided we start from the right data $\{f(n);\, n\in\mathbb{Z}\}$ or $\{f^{(l)}(n);\, n\in\mathbb{Z}, l=0,\cdots, L\}$. In the latter case, we obtain very fast convergence of f *and* its derivatives. However, the theorems do not show how to determine these data or how to estimate smoothness of f given the data $\{k, c_1,\cdots,c_n\}$ specifying (4.1).

In the next section we shall see that the $f(n)$, $n=0$ to $\lceil (k-1)^{-1}N\rceil$ can be related to the eigenvector of eigenvalue 1 of a particular matrix constructed from the coefficients c_n. If this eigenvalue is nondegenerate, then this provides a way to determine the $f(n)$. Similarly, nondegenerate eigenvectors of this matrix, corresponding to the eigenvalue k^{-l}, are linked to the $f^{(l)}(n)$. We shall also see how this matrix provides an upper bound for the regularity of f; more subtle matrix techniques in part II will lead to more precise regularity estimates.

There exists another iterative scheme that is often used for the construction of f. The jth approximation function f_j in this scheme is also a spline function with knot set $2^{-j}\mathbb{Z}$, and $f_{j+1}=Vf_j$, but the initial function f_0 is different. It is a continuous, piecewise linear spline, with $f_0(0)=1$, $f_0(n)=0$ for $n\neq 0$. The advantage of this choice for f_0 is that it results in a "local" algorithm called the *cascade algorithm.* We check (see, e.g., Daubechies (1988)) that, for $0\leqq l<k$,

$$f_j(k^{-j}(km+l))=\sum_n c_{l+kn}f_{j-1}(k^{-j+1}(m-n)).$$

This means that the $f_j(k^{-j}n)$ can be computed by using only the values of f_{j-1} in a small neighborhood of $k^{-j}n$; more precisely, $f_j(k^{-j}n)$ is determined by the $f_{j-1}(k^{-j+1}l)$ with $k^{-j}(n-N)\leqq k^{-(j-1)}l\leqq k^{-j}n$. This is quite unlike the previous scheme, where $f_j(k^{-j}n)$ was computed from the $f_{j-1}(k^{-j-1}n-m)$, $0\leqq m\leqq N$. We remark that in general

$f_0(n) = \delta_{n0}$ does not satisfy the two-scale difference equation (4.1) restricted to $\mathbb{Z}$. It does so only if all but one of the coefficients c_{kn} (with index of a multiple of k) vanish, $c_{kn} = \delta_{n0}$. (We suppose here that $c_n = 0$ for $n < N_1$, or $n > N_2$, where N_1 need not be equal to zero. We can shift this to the standard situation $c_n = 0$ for $n < 0$ or $n > N$; in this case we would have $c_{kn+n_0} = \delta_{n0}$ for some n_0, and we would choose $f_0(n) = \delta_{nn_0}$ correspondingly.) In this case the cascade algorithm corresponds to an *interpolating subdivision scheme* (Chaiken (1974), Dyn, Gregory, and Levin (1987), (1989), (1990), Micchelli (1986), Micchelli and Prautzsch (1987a), (1987b), (1989)): at every level j, the function f_j coincides with f_{j-1} at the knots of f_{j-1}, i.e.,

$$f_j(k^{-j+1}n) = f_{j-1}(k^{-j+1}n);$$

the intermediate values $f_j(k^{-j}(kn+l))$, $0 < l < k$, are computed by an appropriate interpolation procedure (determined by the c_n). The "local" aspect of the cascade algorithm makes subdivision schemes of interest for the construction of curves and surfaces. In Daubechies (1988) the same scheme was called the "graphical" construction algorithm.

A drawback of the cascade algorithm is that it does not always converge, even when a continuous solution to the two-scale difference equation exists. An example is

$$f(x) = \tfrac{1}{2}f(2x-3) + f(2x) + \tfrac{1}{2}f(2x+3).$$

This equation corresponds to a subdivision scheme. It has a continuous solution with support $[-3, 3]$, namely,

$$f(x) = \begin{cases} 1 - |x|/3, & |x| \leqq 3, \\ 0 & \text{otherwise.} \end{cases}$$

The cascade algorithm converges to this solution in the sense of distributions, but not in $C^0(\mathbb{R})$: indeed $f_n(1) = 0$ for all n. In the special case of interpolating subdivision schemes ($c_{km} = \delta_{m0}$), Dyn and Levin (1989) give necessary and sufficient conditions to ensure convergence of the cascade algorithm. Daubechies (1988) lists a different set of sufficient conditions for convergence of the cascade algorithm.

5. Lattice two-scale difference equations: global regularity of compactly supported solutions. Assume that the lattice two-scale difference equation

$$f(x) = \sum_{n=0}^{N} c_n f(kx-n), \tag{5.1}$$

with $\Delta = (1/k)\sum_{n=0}^{N} c_n = 1$, has a nontrivial L^1-solution, necessarily of compact support. The regularity of this function can be bounded above purely in terms of its support width.

THEOREM 5.1. *Given a lattice two-scale difference equation,* $f(x) = \sum_{n=0}^{N} c_n f(kx-n)$ *with* $\Delta = 1$. *Let* N_0 *be the largest integer strictly smaller than* $N/(k-1)$, *and define* M *to be the* $N_0 \times N_0$ *matrix*

$$\mathsf{M}_{ij} = c_{ki-j}, \qquad i, j = 1, \cdots, N_0. \tag{5.2}$$

If there exists a nontrivial L^1*-solution* f *which is in* $C^m(\mathbb{R})$, *then* $\{1, k^{-1}, \cdots, k^{-m}\} \subset$ *spectrum* (M). *In particular,*

$$m < \frac{N}{k-1} - 1. \tag{5.3}$$

Proof. (1) Since f is continuous, and support $(f) \subset [0, N/(k-1)]$, it follows that $f(l) = 0$ for $l \leqq 0$ and $l > N_0$. Define $\mathsf{v}^0 \in \mathbb{R}^{N_0}$ by

$$\mathsf{v}_j^0 = f(j), \qquad j = 1, \cdots, N_0.$$

Substituting $x = j$, with $j = 1, \cdots, N_0$, into equation (5.1) leads to

$$\mathsf{v}^0 = \mathsf{M}\mathsf{v}^0,$$

where M is defined by (5.2).

(2) On the other hand, it is clear that $\mathsf{v}^0 \neq 0$. Indeed, if $\mathsf{v}^0 = \mathbf{0}$, i.e. $f(j) = 0$ for all $j \in \mathbb{Z}$, then $f(k^{-l}m) = 0$ would follow, for all $l \in \mathbb{N}$, $m \in \mathbb{Z}$, by applying (5.1). By continuity this would imply $f \equiv 0$. Since $f \neq 0$, we have $\mathsf{v}^0 \neq 0$. Consequently, 1 is an eigenvalue of M.

(3) Similarly we define, for $l \leqq m$, $\mathsf{v}^l \in \mathbb{R}^{N_0}$ by

$$\mathsf{v}_j^l = f^{(l)}(j), \qquad j = 1, \cdots, N_0,$$

where $f^{(l)}$ denotes the lth derivative of f. Differentiating (5.1) l times, and substituting $x = 1, \cdots, N_0$ leads to

$$\mathsf{v}^l = k^l \mathsf{M}\mathsf{v}^l.$$

Again $\mathsf{v}^l = 0$ would imply $f^{(l)} \equiv 0$, hence $f^{(l-1)} \equiv$ constant. Since $f^{(l-1)}$ has compact support, $f^{(l-1)} \equiv 0$ would follow. By induction this would imply that $f \equiv 0$. Since $f \neq 0$, $\mathsf{v}^l \neq 0$, and k^{-l} is an eigenvalue of M for $0 \leqq l \leqq m$.

(4) $f \in C^m(\mathbb{R})$ implies that the $N_0 \times N_0$ matrix M has $m+1$ eigenvalues. Hence $m \leqq N_0 - 1 < N/(k-1) - 1$. □

Remarks. (1) The bound (5.3) of Theorem 5.1 cannot be improved. For $N = (k-1)L$ there exist $\{c_n; n = 0, 1, \cdots, N\}$ such that the $(L-1)\times(L-1)$ matrix M has exactly the eigenvalues $1, k^{-1}, \cdots, k^{-L+2}$, and such that the corresponding f is in c^{L-2}. One such example is given by

$$p(\xi) = \sum_{n=0}^{(k-1)L} c_n e^{in\xi} = [(1 + e^{i\xi} + \cdots + e^{i(k-1)\xi})/k]^L,$$

leading to $\hat{f}(\xi) = [(1 + e^{i\xi})/\xi]^L$. The function f is a B-spline of degree $L-1$; it is in C^{L-2}. The fact that any C^{n-1}-spline with knot set $\mathbb{Z}$ must have support width greater than or equal to $n+2$ has long been known (cf. Schoenberg (1973, p. 13)).

(2) The condition $\{1, k^{-1}, \cdots, k^{-m}\} \subset$ spectrum (M) is not sufficient to ensure that $f \in C^m$. For $k = 2$, $N = 3$, e.g., all the choices $c_0 = \frac{1}{4} + \lambda$, $c_1 = \frac{3}{4} + \lambda$, $c_2 = \frac{3}{4} - \lambda$, $c_3 = \frac{1}{4} - \lambda$, all other $c_n = 0$, where $\lambda \in \mathbb{R}$ is arbitrary, lead to 2×2 matrices M with the same spectrum, namely, $\{1, \frac{1}{2}\}$. Nevertheless, the regularity of f depends on λ. Using the techniques of part II, we can check that f is continuous if and only if $|\lambda| < \frac{3}{4}$, and that $f \in C^1$ if and only if $|\lambda| < \frac{1}{4}$. For $\lambda = \frac{1}{4}$, e.g., we find $p(\xi) = (1 + e^{i\xi})^2/4$; hence $f(x) = x$ for $0 \leqq x \leqq 1$, $2 - x$ for $1 \leqq x \leqq 2$, zero otherwise, which is clearly not in C^1.

(3) It follows from the proof that, provided that they are nondegenerate, the eigenvectors of M with eigenvalue k^{-l} determine the $f^{(l)}(n)$, up to normalization. It is not a priori obvious how to choose these $m+1$ different normalizations (one for each l) in a coherent way. In part II we shall see how this can be done, modulo some restrictions on the c_n.

6. Examples.

6.1. The de Rham function. The de Rham function is a classical example of a continuous nowhere-differentiable function. Like many such examples, it is defined as the limit of successive approximations.

Define

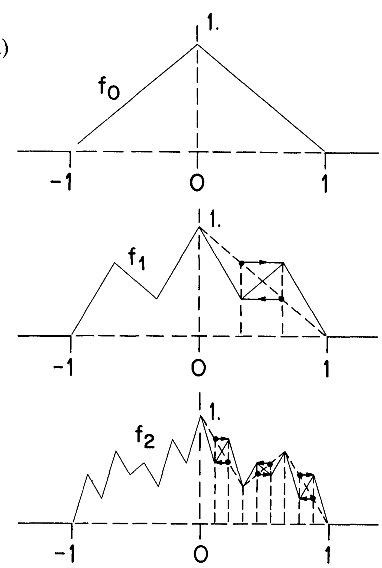
$$f_0(x)=\begin{cases}1+x, & -1\leqq x\leqq 0,\\ 1-x, & 0\leqq x\leqq 1,\\ 0, & |x|\geqq 1.\end{cases}$$

Clearly f_0 is piecewise linear; its restriction to the intervals $[m, m+1]$, $m\in\mathbb{Z}$, is linear. The next function f_1 in the approximation scheme is constructed as follows: f_1 is again piecewise linear, with its restriction to the intervals $[m/3,(m+1)/3]$ linear, for all $m\in\mathbb{Z}$. The nodes of f_1 are given by $f_1(m)=f_0(m)$, $f_1(m+\frac{1}{3})=f_0(m+\frac{2}{3})$, $f_1(m+\frac{2}{3})=f_0(m+\frac{1}{3})$, for all $m\in\mathbb{Z}$. Graphically, this corresponds to splitting every interval on which f_0 is linear into three equal parts, exchanging the values at f_0 at the two interior points, and linearly interpolating between the nodes obtained in this way (see Fig. 1). Exactly the same procedure is then repeated to obtain f_{j+1} from f_j, for all $j\in\mathbb{N}$. The resulting f_j are piecewise linear, with linear restrictions to the intervals $[m3^{-j}, (m+1)3^{-j}]$, for all $m\in\mathbb{Z}$. Geometrically it is clear that this process converges pointwise to a continuous limit function f.

It can be checked fairly easily that $f_{n+1}=Vf_n$ where

$$Vf(x)=f(3x)+\tfrac{1}{3}[f(3x+1)+f(3x-1)]+\tfrac{2}{3}[f(3x+2)+f(3x-2)].$$

(a)



(b)

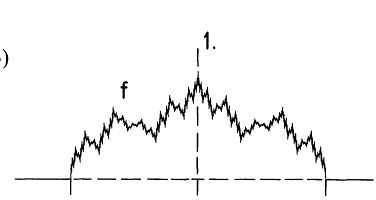


FIG. 1. (a) *The first three approximations f_0, f_1, f_2 to the de Rham function.* (b) *The de Rham function.* (*Note. We have plotted f_8 rather than f. At the scale of the figure, they are indistinguishable.*)

As the pointwise limit of the f_j, the de Rham function f satisfies the two-scale difference equation

(6.1) $$f(x)=f(3x)+\tfrac{1}{3}[f(3x+1)+f(3x-1)]+\tfrac{2}{3}[f(3x+2)+f(3x-2)].$$

The corresponding trigonometric polynomial is given by

$$p(\xi)=e^{-2i\xi}[(1+e^{i\xi}+e^{2i\xi})/3][(2-e^{i\xi}+2e^{2i\xi})/3];$$

it is not clear how to deduce the continuity of f from this expression for p! In part II we shall use a time-domain method to prove that f is Hölder continuous, with exponent $\gamma=1-\ln 2/\ln 3=.36907\cdots$ but is nowhere differentiable. The method of part II also allows us to analyze local properties of f, to show that there exist fractal sets with nonzero Hausdorff dimension, but zero Lebesgue measure, on which f is "almost" differentiable, in the sense that the local Hölder exponent can be chosen arbitrarily close to 1. (The choice of the fractal set depends on the desired Hölder exponent.)

A variant on the de Rham function is obtained by choosing $k=3$, $c_0=1$, $c_1=c_{-1}=\frac{1}{2}-\alpha$, $c_2=c_{-2}=\frac{1}{2}+\alpha$, all other $c_n\equiv 0$. The corresponding f_j^α and f^α are plotted in Fig. 2; for $\alpha=\frac{1}{6}$ we obviously revert to the de Rham case. The analysis of part II will show

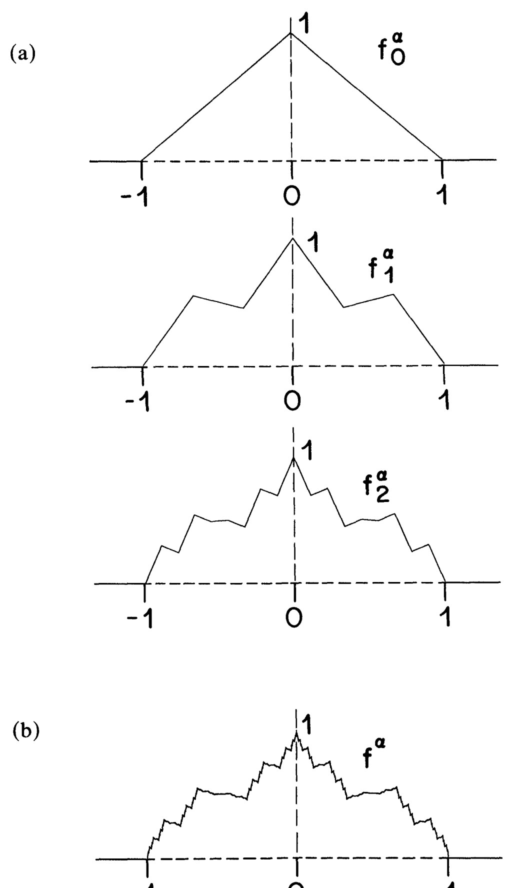


FIG. 2. (a) *The first three approximations* f_j^α, $j=0,1,2$ *for the generalized de Rham function corresponding to* $\alpha=\frac{1}{12}$. (b) *The generalized de Rham function* f^α *itself.*

that for sufficiently small α, $\alpha(1+2\alpha)^2<\frac{2}{27}$, or $\alpha<.0492\cdots$, the resulting function f^{α} is Lipschitz almost everywhere.

Note that for any α we have $N=4$, hence $N_0=1$, so that the matrix M reduces to the scalar 1. It therefore follows immediately from Theorem 5.1 that f and f^{α} cannot possibly be C^1, since then M would have at least the two eigenvalues 1 and $\frac{1}{2}$.

6.2. The Lagrange interpolation functions of Deslauriers and Dubuc. These functions are obtained by choosing an integer, $k>1$, called the "base" of the interpolation scheme, and an even integer M, $M\geqq 2$, called the "number of nodes," in the language of Deslauriers and Dubuc (1989). The interpolation function is then defined by the recursive process

$$f[k^{-j}(km+n)]=\sum_{m'=-M/2+1}^{M/2}\beta_{m',n}f[k^{-(j-1)}(m+m')],$$

where

$$\sum_{m'=-M/2+1}^{M/2}\beta_{m',n}=1 \quad \text{for all } n=1,\cdots,k-1. \tag{6.2}$$

This corresponds to a two-scale equation of the type

$$f(x)=f(kx)+\sum_{m=-M/2}^{M/2-1}\sum_{m=1}^{k-1}\alpha_{m,n}f(kx-km-n)$$

where $\alpha_{mn}=\beta_{-m,n}$. The $\beta_{m,n}$, or α_{mn}, are determined by (6.2), by the requirement that $p(\xi)$ be divisible by as many factors $[1+e^{i\xi}+\cdots+e^{i(k-1)\xi}]$ as possible, and by the symmetry condition $\beta_{-l,n}=\beta_{l+1,k-n}$, $0\leqq l\leqq M/2$, $n=1,\cdots,k-1$. For base 2, with four nodes ($k=2$, $M=4$), this leads to the two-scale difference equation

$$f(x)=f(2x)+\tfrac{9}{16}[f(2x+1)+f(2x-1)]-\tfrac{1}{16}[f(2x+3)+f(2x-3)], \tag{6.3}$$

corresponding to

$$\begin{aligned}p(\xi)&=e^{-3i\xi}[(1+e^{i\xi})/2]^4[-\tfrac{1}{2}+2e^{i\xi}-\tfrac{1}{2}e^{2i\xi}]\\&=(\cos\xi/2)^4(2-\cos\xi).\end{aligned}$$

Using $\sup_{\xi}|2-\cos\xi|=3$, we obtain $|\hat{f}(\xi)|\leqq C|\xi|^{-4+\log_2 3}$ (see, e.g., Lemma 3.2 in Daubechies (1988)), from which it follows that $f\in C^1$. We can bound the regularity of $f(x)$ by Theorem 5.1. We have $N=6$, $k=2$, so that $N_0=5$. The spectrum of the 5×5 matrix M is in this case $\{1,\frac{1}{2},\frac{1}{4},\frac{1}{8}\}$ where the eigenvalue $\frac{1}{4}$ has multiplicity 2. By Theorem 5.1 we know therefore that f can be at most C^3. In fact, however, f is not even C^2, as is shown in Deslauriers and Dubuc (1989), using the infinite product formula for $\hat{f}$ together with the special property that $p(\xi)\geqq 0$. They show that f is "almost" C^2, in the sense that f' is Hölder continuous with exponent $1-\varepsilon$ (ε arbitrarily small), but not C^1. In Dubuc (1986) the sharper estimate $|f(x)-f(x+t)|\leqq C|t|\log(1/|t|)$ is proved, for small enough t. This same example was also studied by Dyn, Gregory, and Levin (1987), with more general weights in (6.3) ($\frac{1}{2}+w$, w instead of $\frac{9}{16}$, $\frac{1}{16}$). For the parameters fixed as in (6.3), their results are slightly weaker than Dubuc's. For a thorough and detailed analysis of this example we refer to Dubuc (1986), Dyn, Gregory, and Levin (1987), or to part II.

6.3. Orthonormal bases of compactly supported wavelets. A family of wavelets is generated by translating and dilating one single function,

$$\psi_{jk}(x)=2^{-j/2}\psi(2^{-j}x-k), \qquad j,k\in\mathbb{Z}.$$

For some choices of ψ, the family ψ_{jk} constitutes an orthonormal basis of $L^2(\mathbb{R})$. One such choice is

$$\psi(x)=\begin{cases} 1, & 0\leqq x<\frac{1}{2}, \\ -1, & \frac{1}{2}\leqq x<1, \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding orthonormal basis is well known; it is called the Haar basis, and provides an unconditional basis for all L^p-spaces, $1<p<\infty$. Recently some other, more interesting choices for ψ have been found. The first one was constructed by Stromberg (1982); later Meyer (1985/86) constructed independently another wavelet basis, which was extended to higher dimensions by Lemarié and Meyer (1986). In the Meyer construction $\hat{\psi}$ is C^∞ and compactly supported; the basis $\{\psi_{jk}\}$ is not only an orthonormal basis for $L^2(\mathbb{R})$, but also an unconditional basis for all the L^p-spaces $(1<p<\infty)$, the Sobolev spaces, the Besov spaces, etc. Later Battle (1987) and Lemarié (1987) constructed other orthonormal bases of wavelets, based on ψ which have faster (exponential) decay; their examples are K times continuously differentiable (K arbitrarily large, but finite). Mallat (1989) and Meyer (1986), (1990) devised a scheme into which all these constructions fit naturally, which they call *multiresolution analysis.* Finally, Daubechies (1988) constructed orthonormal bases of wavelets generated by *compactly supported* ψ which are K times differentiable.

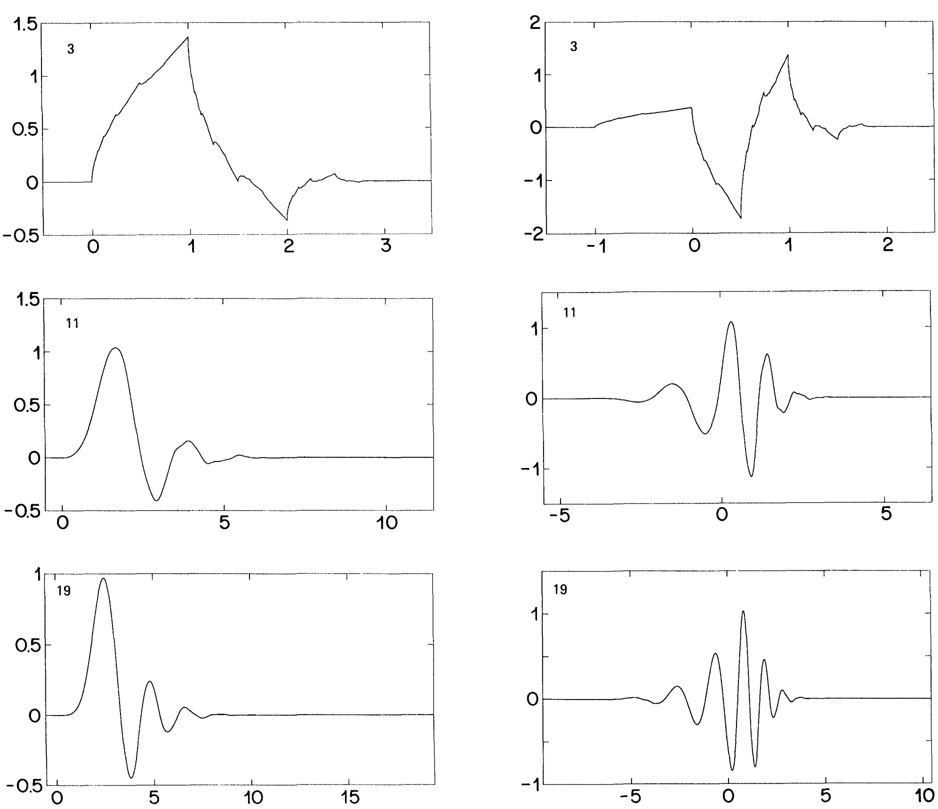


FIG. 3. *Some examples of orthonormal wavelet bases with compact support constructed in Daubechies* (1988). *In every case both* ϕ *and* ψ *are plotted. The number of nonvanishing* c_n *is, respectively,* 4, 12, *and* 20, *corresponding to support widths of, respectively,* 3, 11, *and* 19.

A typical construction of an orthonormal basis of wavelets uses an auxiliary function ϕ such that

$$\phi(x) = \sum c_n \phi(2x - n). \tag{6.4}$$

Provided the $\phi(x-n)$ are an orthonormal set, the function ψ is then given by

$$\psi(x) = \sum_n (-1)^n c_{n+1} \phi(2x+n).$$

(If the functions $\phi(x-n)$ are not orthonormal, we first construct $\tilde{\phi}$ by $\hat{\tilde{\phi}}(\xi) = \hat{\phi}(\xi) \times (\sum_{m\in\mathbb{Z}} |\hat{\phi}(\xi + 2\pi m)|^2)^{-1/2}$; the $\tilde{\phi}(x-n)$ are orthonormal, and satisfy an equation of type (6.4), with different $\tilde{c}_n$; for more details see Daubechies (1988) or Mallat (1989)).

A construction of ϕ using only finitely many c_n results in a compactly supported ϕ (see § 2), and therefore a compactly supported ψ. As a finite linear combination of translated and dilated versions of ϕ, ψ has the same regularity as ϕ. It follows that a good understanding of the regularity of solutions of finite two-scale difference equations is important in the construction of orthonormal bases of compactly supported wavelets. The examples constructed in Daubechies (1988) have the property that their support width increases linearly with their regularity. This is illustrated by Fig. 3, which shows the pairs ϕ, ψ for support widths 3, 11, and 19, respectively. It is clear that ϕ, ψ become more regular as their width increases; Daubechies (1988) showed that there exists $\mu > 0$ such that

$$\phi_N, \psi_N \in C^{\mu N} \quad \text{where } |\text{supp } \phi_N| = |\text{supp } \psi_N| = N.$$

The question then arose whether this linear increase of the support width was necessary. This question is now answered affirmatively by Theorem 5.1: if $\phi \in C^K$, then $|\text{supp } \phi| \geqq K+2$. This also provides a simple proof for the (known) fact that it is impossible to construct wavelet bases generated by a compactly supported C^∞-function ϕ.

Appendix.

PROPOSITION. *Suppose f is a compactly supported function in* $\text{Lip}^{L,\alpha}$. *Define functions f_j by*:

(1) *On every interval* $[n2^{-j}, (n+1)2^{-j}]$, *f_j is a polynomial of degree* $2L+1$.

(2) *f_j is in C^L and*

$$f_j^{(l)}(n2^{-j}) = f^{(l)}(n2^{-j}) \quad \text{for } l = 0, \cdots, L, \quad n \in \mathbb{Z}.$$

Then $\|f^{(l)} - f_j^{(l)}\|_{L^\infty} \leqq C2^{-j(L-l+\alpha)}$ *for* $l = 0, \cdots, L$, *and for some C independent of j and l.*

Proof. (1) Choose $x \in \text{support}(f)$, $j \in \mathbb{N}$ arbitrary. Find n so that $2^{-j}n \leqq x \leqq 2^{-j}(n+1)$. Then

$$\begin{aligned} |f^{(l)}(x) - f_j^{(l)}(x)| \leqq & \left| f^{(l)}(x) - \sum_{k=l}^{L} \frac{1}{(k-l)!} f^{(k)}(2^{-j}n)(x - 2^{-j}n)^{(k-l)} \right| \\ & + \left| f_j^{(l)}(x) - \sum_{k=l}^{L} \frac{1}{(k-l)!} f^{(k)}(2^{-j}n)(x-2^{-j}n)^{(k-l)} \right|. \end{aligned} \tag{A.1}$$

Since $f \in \text{Lip}^{L,\alpha}$ and support (f) is finite, the first term is bounded by $C|x - 2^{-j}n|^{\alpha+(L-l)} \leqq C2^{-j(\alpha+L-l)}$ with C independent of x or j. It therefore suffices to bound the second term.

(2) On $[2^{-j}n, 2^{-j}(n+1)]$ we have

$$f_j(y) = \sum_{l=0}^{L} \frac{1}{l!} f^{(l)}(2^{-j}n)(y - 2^{-j}n)^l + \sum_{l=0}^{L} \frac{1}{(L+1+l)!} a_{n,l}^j (y - 2^{-j}n)^{L+1+l}$$

with the $a_{n,l}^j$ determined by the $L+1$ equations, $0 \leqq m \leqq L$,

$$\sum_{l=0}^{L} \frac{1}{(L+1+l-m)!} a_{n,l}^j 2^{-j(L+1+l-m)} = f^{(m)}(2^{-j}(n+1)) - \sum_{l=m}^{L} \frac{1}{(l-m)!} f^{(l)}(2^{-j}n) 2^{-j(l-m)},$$

or

$$\sum_{l=0}^{L} \frac{1}{(L+1+l-m)!} a_{n,l}^j 2^{-j(l+1)} = b_{n,m}^j \tag{A.2}$$

where

$$b_{n,m}^j = 2^{-j(L-m)} \left[f^{(m)}(2^{-j}(n+1)) - \sum_{l=m}^{L} \frac{1}{(l-m)!} f^{(l)}(2^{-j}n) 2^{-j(l-m)} \right]$$

is bounded, uniformly in n, by $C2^{-j\alpha}$ because $f \in \mathrm{Lip}^{L,\alpha}$ and f is compactly supported. It follows, by inverting the system (A.2), that

$$|a_{n,l}^j 2^{-j(l+1)}| \leqq C2^{-j\alpha}.$$

(3) Consequently,

$$\begin{aligned} &\left| f_j^{(l)}(x) - \sum_{k=1}^{L} \frac{1}{(k-l)!} f^{(k)}(2^{-n}j)(x - 2^{-j}n)^{(k-l)} \right| \\ &= \left| \sum_{k=0}^{L} \frac{1}{(L+1+k-l)!} a_{n,k}^j (x - 2^{-j}n)^{L+1+k-l} \right| \\ &\leqq \sum_{k=0}^{L} \frac{1}{(L+1+k-l)!} C 2^{j(k+1)} 2^{-j\alpha} 2^{-j(L+1+k-l)} \\ &\leqq C2^{-j(L-l+\alpha)}. \end{aligned}$$

Hence (A.1) $\leqq C2^{-j(L-l+\alpha)}$ for all x, with C independent of j, l, or x, and the proposition is proved. □

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