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ON THE INSTABILITY OF ARBITRARY BIORTHOGONAL WAVELET PACKETS*

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Abstract. Starting from a multiresolution analysis and the corresponding orthonormal wavelet basis, Coifman and Meyer have constructed wavelet packets, a library from which many different orthonormal bases can be picked. This paper proves that when the same procedure is applied to biorthogonal wavelet bases, not all the resulting wavelet packets lead to Riesz bases for $L^2(\mathbb{R})$.

Key words. wavelet packets, biorthogonal wavelets

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1. Short review of orthonormal wavelet bases.

An orthonormal basis of wavelets $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$, $j, k \in \mathbb{Z}$, associated with a multiresolution analysis, is completely determined by a 2π-periodic function $m_0(\xi)$. More precisely,

$$\hat{\psi}(\xi) = e^{-i\xi/2}\overline{m_0\left(\frac{\xi}{2}+\pi\right)}\hat{\phi}\left(\frac{\xi}{2}\right), \tag{1.1}$$

with

$$\hat{\phi}(\xi) = (2\pi)^{-1/2}\prod_{j=1}^{\infty} m_0(2^{-j}\xi). \tag{1.2}$$

Here ^ denotes the Fourier transform, normalized by

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int dx e^{-ix\xi} f(x).$$

Conversely, given a 2π-periodic function m_0, one can define (1.1) and (1.2); if m_0 satisfies a few conditions, then the resulting ψ will generate an orthonormal wavelet basis. Which conditions? Let us assume that m_0 is continuous (which is the case in all useful examples). Then in order for (1.2) to converge, we need $m_0(0) = 1$. If moreover $|m_0(\xi) - 1| \leq C|\xi|^{\alpha}$ for some $\alpha > 0$, then (1.2) converges uniformly on compact sets. (This is not really necessary, but satisfied in all examples of even the remotest interest.) Furthermore, orthonormality of the $\psi_{j,k}$ implies that

$$|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1 \tag{1.3}$$

(see Mallat [14]). This is not sufficient to ensure orthonormality of the $\psi_{j,k}$, however; to guarantee this orthonormality we need one more (necessary and sufficient) condition on m_0, of a more technical nature: there should exist a compact set K, congruent with $[-\pi, \pi]$ modulo 2π, such that

$$\inf_{\xi\in K}\inf_{n\geq 1}|m_0(2^{-n}\xi)| > 0. \tag{1.4}$$

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If all these conditions on m_0 are verified, then the $\psi_{j,k}$'s do indeed constitute an orthonormal basis for $L^2(\mathbb{R})$. Note that $m_0(0) = 1$ automatically ensures that $|m_0(2^{-n}\xi)| > \frac{1}{2}$ for sufficiently large n and all $\xi \in K$ (because K is compact), so that (1.4) is a constraint for only finitely many values of n. See Cohen [4] or Cohen [5] for a proof of the necessity and sufficiency of this last condition; the condition can also be recast in other forms [13], [5], [6]. A consequence of (1.4) is that (see Cohen [5])

$$\inf_{\xi \in K} |\hat{\phi}(\xi)| > 0. \tag{1.5}$$

The orthonormal basis $\{\psi_{j,k}; j, k \in \mathbb{Z}\}$ generated by ψ can be interpreted within the framework of a *multiresolution analysis* (see Mallat [14], Meyer [15]). Let V_0 be the space spanned by the functions $\phi(x-k)$, $k \in \mathbb{Z}$ (which are also orthonormal). Define V_j to be the space obtained by dilating V_0 by 2^j,

$$f \in V_j \Longleftrightarrow f(2^{-j}\cdot) \in V_0;$$

an orthonormal basis of V_j is given by $\{\phi_{j,k};\ k \in \mathbb{Z}\}$, with $\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)$. Then

$$\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots,$$

and

$$\bigcap_{j\in\mathbb{Z}} V_j = \{0\}, \qquad \overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}).$$

Let W_j be the orthogonal complement in V_{j+1} of V_j. Then $\{\psi_{j,k};\ k \in \mathbb{Z}\}$ is an orthonormal basis in W_j, and

$$\bigoplus_{j=-\infty}^{\infty} W_j = L^2(\mathbb{R}). \tag{1.6}$$

This is the standard decomposition of $L^2(\mathbb{R})$ into different "layers" of wavelets with resolution 2^{-j}. One can also choose to use only reasonably fine scale wavelets, and to lump the coarser aspects together into one space, corresponding to the decomposition

$$L^2(\mathbb{R}) = V_0 \oplus \left(\bigoplus_{j=0}^{\infty} W_j \right). \tag{1.7}$$

2. Orthonormal wavelet packets. Given a 2π-periodic function m_0 which satisfies all the conditions in §1, one can define many other orthonormal bases, corresponding to decompositions of $L^2(\mathbb{R})$ different from (1.6) or (1.7). They are all designated by the name "wavelet packets," first defined by Coifman and Meyer; for a discussion of their properties and some applications, see the two papers by Coifman, Meyer, and Wickerhauser in [16]. Their construction can be understood easily by using the following lemma [8].

LEMMA 2.1 (the "splitting trick"). *Suppose that the functions $f_k(x) = f(x-k)$, $k \in \mathbb{Z}$, are orthonormal. Define f^0, f^1 by*

$$\hat{f^\sigma}(\xi) = m_\sigma(\xi/2)\hat{f}(\xi/2), \qquad \sigma = 0, 1,$$

where m_0 *is as above,* $m_1(\xi) = e^{-i\xi}\,\overline{m_0(\xi+\pi)}$. *Then the functions* $f_k^0(x) = \frac{1}{\sqrt{2}} f^0\left(\frac{x}{2}-k\right)$, $f_k^1(x) = \frac{1}{\sqrt{2}} f^1\left(\frac{x}{2}-k\right)$, $k \in \mathbb{Z}$, *constitute an orthonormal basis for* $E = \overline{\mathrm{Span}\{f_k\}}$.

Remark. Note that with this notation convention, $f_0^\sigma(x) = \frac{1}{\sqrt{2}} f^\sigma\left(\frac{x}{2}\right)$ and not $f^\sigma(x)$.

Proof.

1. Since

$$\langle f_k, f_\ell\rangle = \int d\xi |\hat{f}(\xi)|^2 e^{i(k-\ell)\xi}$$
$$= \int_0^{2\pi} d\xi e^{i(k-\ell)\xi} \sum_{m\in\mathbb{Z}} |\hat{f}(\xi+2\pi m)|^2,$$

orthonormality of the f_k is equivalent to $\sum_{m\in\mathbb{Z}} |\hat{f}(\xi+2\pi m)|^2 = \frac{1}{2\pi}$ almost everywhere.

2. Similarly,

$$\langle f_k^0, f_\ell^0\rangle = 2\int d\xi |\hat{f}^0(2\xi)|^2 e^{2i(k-\ell)\xi}$$
$$= 2\int_0^{\pi} d\xi e^{2i(k-\ell)\xi} \sum_{m\in\mathbb{Z}} |\hat{f}^0(2\xi+2\pi m)|^2.$$

Splitting the sum over m into even and odd m leads to

$$2\sum_{m\in\mathbb{Z}} |\hat{f}^0(2\xi+2\pi m)|^2 = 2|m_0(\xi)|^2 \sum_n |\hat{f}(\xi+2\pi n)|^2$$
$$+ 2|m_0(\xi+\pi)|^2 \sum_n |\hat{f}(\xi+\pi+2\pi n)|^2 = \frac{1}{\pi},$$

proving that the f_k^0 are orthonormal. Orthonormality of the f_k^1 is proved analogously, as well as orthogonality of f_k^0 and f_ℓ^1.

3. On the other hand, if $\sum_k c_k f_k \perp f_\ell^\sigma$ for all $\ell \in \mathbb{Z}$ $\sigma = 0, 1$, then

$$0 = \left\langle f_\ell^\sigma, \sum_k c_k f_k \right\rangle = \sqrt{2} \sum_k \bar{c}_k \int_0^{2\pi} d\xi e^{i(k-2\ell)\xi} m_\sigma(\xi),$$

i.e. $c(\xi) = \sum_k c_k e^{-ik\xi}$ is orthogonal (in $L^2([0,2\pi])$) to the $e^{-i2\ell\xi} m_\sigma(\xi), \ell \in \mathbb{Z}$, $\sigma = 0, 1$, implying

$$\overline{c(\xi)} m_\sigma(\xi) + \overline{c(\xi+\pi)} m_\sigma(\xi+\pi) = 0 \quad \text{a.e.}, j = 0, 1.$$

Multiplying with $\overline{m_\sigma(\xi)}$, and adding the two equations gives

$$0 = \overline{c(\xi)}\left[|m_0(\xi)|^2 + m_0(\xi+\pi)|^2\right]$$
$$+ \overline{c(\xi+\pi)}\left[m_0(\xi+\pi)\overline{m_0(\xi)} - \overline{m_0(\xi)} m_0(\xi+\pi)\right] = \overline{c(\xi)},$$

proving that $\sum_k c_k f_k = 0$, so that the $\{f_\ell^\sigma; \ell \in \mathbb{Z}, \sigma = 0, 1\}$ span all of E. □

Modulo appropriate dilations, the splitting trick can of course also be applied to spaces with an orthonormal basis generated by the regularly spaced translates $f(x-ak)$, $k \in \mathbb{Z}$ of a single function, even if $a \neq 1$.

If the splitting trick is applied to V_0, the space with orthonormal basis $\{\phi_{0,k}; k \in \mathbb{Z}\}$, where $f = \phi$, then the corresponding functions f^0, f^1 are exactly $f^0(x) = \phi(x)$ and $f^1(x) = \psi(x)$; it easily follows that the splitting of V_0 into the spaces generated by the f_k^0 on one hand, the f_k^1 on the other hand, is exactly $V_0 = V_{-1} \oplus W_{-1}$. One can, therefore, view the transition from (1.7) to (1.6) as a result of infinitely many successive splittings, where at every step the V_j space with smallest index gets split into two.

Starting with (1.7) one can of course choose to apply the splitting trick to many subspaces other than V_0, leading to many different orthonormal bases. Every one of the functions generated in this way can be labelled by an overall dilation J, a translation k, and a sequence ϵ consisting of only ones and zeros, and ending in a tail of all zeros. Concretely,

$$f_{J,k;\epsilon}(x) = 2^{J/2}\psi_\epsilon(2^J x - k)$$

with

$$\hat{\psi}_\epsilon(\xi) = (2\pi)^{-1/2}\prod_{j=1}^{\infty} m_{\epsilon_j}(2^{-j}\xi);$$

if $j_{\max}$ is the largest index for which $\epsilon_{j_{\max}} = 1$, this can be rewritten as

$$\hat{\psi}_\epsilon(\xi) = \left[\prod_{j=1}^{j_{\max}} m_{\epsilon_j}(2^{-j}\xi)\right] \hat{\phi}(2^{-j_{\max}}\xi).$$

The function $f_{J,k;\epsilon}$ is the result of $j_{\max} - 1$ splittings of $W_{j_{\max}+J-1}$. For every fixed choice of a sequence of splittings the result is an orthonormal wavelet packet basis. In applications to signal analysis, one can use entropy estimates to find the "best" basis (Coifman and Wickerhauser [11]).

An important special case is where each W_j space is split exactly j times. The resulting orthonormal basis functions are the integer translates of all the ψ_ϵ, with ϵ ranging over all possible sequences of zeros and ones, with a tail of all zeros. This orthonormal basis is, of all the wavelet packet bases, the closest to a windowed Fourier transform.

3. Biorthogonal wavelet bases. There exist orthonormal wavelet bases with compactly supported ψ and ϕ. The 2π-periodic function m_0 is then a trigonometric polynomial. By imposing a factorization of the type

$$m_0(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^N Q(\xi),$$

one can construct ϕ and ψ with arbitrarily high degree of smoothness [12]. One inconvenience of these compactly supported orthonormal wavelets is that they are not symmetric. One can restore symmetry by relaxing the orthonormality requirement. In this case one works with *two* 2π-periodic functions, m_0 and $\tilde{m}_0$, satisfying

$$m_0(\xi)\overline{\tilde{m}_0(\xi)} + m_0(\xi+\pi)\overline{\tilde{m}_0(\xi+\pi)} = 1. \tag{3.1}$$

There are similarly two pairs of scaling functions and wavelets, defined by

$$\text{(3.2)} \qquad \hat{\phi}(\xi) = (2\pi)^{-1/2} \prod_{j=1}^{\infty} m_0(2^{-j}\xi), \qquad \hat{\tilde{\phi}}(\xi) = (2\pi)^{-1/2} \prod_{j=1}^{\infty} \tilde{m}_0(2^{-j}\xi)$$

$$\hat{\psi}(\xi) = e^{-i\xi/2}\overline{\tilde{m}_0\left(\tfrac{\xi}{2}+\pi\right)}\hat{\phi}\left(\tfrac{\xi}{2}\right), \qquad \hat{\tilde{\psi}}(\xi) = e^{-i\xi/2}\overline{m_0\left(\tfrac{\xi}{2}+\pi\right)}\hat{\tilde{\phi}}\left(\tfrac{\xi}{2}\right).$$

The $\psi_{j,k}$ and $\tilde{\psi}_{j,k}$ now constitute dual Riesz bases; both ψ and $\tilde{\psi}$ can be symmetric (or antisymmetric) and have compact support (Cohen, Daubechies, and Feauveau [9]). (Note that there also exists a different scheme of biorthogonal Riesz bases of wavelets, in which ψ and $\tilde{\psi}$ are symmetric or antisymmetric, and one of them is compactly supported, while the other is not. See Chui and Wang [3] and Chui [1].)

This corresponds to m_0 and $\tilde{m}_0$ which are trigonometric polynomials. We shall restrict ourselves to this case. In order for the whole construction to work, we need, of course, to impose again some conditions on m_0 and $\tilde{m}_0$. First, we need $m_0(0) = 1 = \tilde{m}_0(0)$; the infinite products in (3.2) then converge uniformly on compact sets. We also need more technical conditions. One of them is similar to the orthonormal case; i.e., we need that for some compact set K, congruent to $[-\pi, \pi]$ modulo 2π,

$$\text{(3.3)} \qquad \inf_{\xi \in K} \inf_{n>0} |m_0(2^{-n}\xi)| > 0$$

and

$$\inf_{\xi \in K} \inf_{n>0} |\tilde{m}_0(2^{-n}\xi)| > 0.$$

Another condition concerns the spectral radius of two matrices derived from m_0 and $\tilde{m}_0$ (see Cohen and Daubechies [7]). One consequence of (3.3) (the only one we shall use here) is that

$$\inf_{\xi \in K} |\hat{\phi}(\xi)| > 0, \qquad \inf_{\xi \in K} |\hat{\tilde{\phi}}(\xi)| > 0$$

(see Cohen, Daubechies and Feauveau [9]).

The two pairs of scaling functions generate two multiresolution hierarchies,

$$\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots$$
$$\cdots \subset \tilde{V}_{-2} \subset \tilde{V}_{-1} \subset \tilde{V}_0 \subset \tilde{V}_1 \subset \tilde{V}_2 \subset \cdots,$$

and the $W_j = \overline{\mathrm{Span}\{\psi_{j,k};\ k \in \mathbb{Z}\}}$, (or $\tilde{W}_j = \overline{\mathrm{Span}\{\tilde{\psi}_{j,k};\ k \in \mathbb{Z}\}}$) are still complement spaces of the V_j in V_{j+1} (or $\tilde{V}_j$ in $\tilde{V}_{j+1}$), though no longer orthogonal complements. The two hierarchies are linked via the property that, for all $j \in \mathbb{Z}$,

$$\tilde{W}_j \perp V_j, W_j \perp \tilde{V}_j.$$

We can again decompose $L^2(\mathbb{R})$ as either

$$\text{(3.4)} \qquad L^2(\mathbb{R}) = \bigoplus_{j \in \mathbb{Z}} W_j$$

or

$$\text{(3.5)} \qquad L^2(\mathbb{R}) = V_0 \oplus \bigoplus_{j=0}^{\infty} W_j,$$

where the direct sums are not sums of orthogonal spaces. If we define Q_j to be the (nonorthogonal) projection operator onto W_j associated with this expansion, then the L^2-norm $\|u\|^2$ is still equivalent with $\sum_j \|Q_j u\|^2$:

LEMMA 3.1. *Let $\psi_{j,k}$, $\tilde{\psi}_{j,k}$ be biorthogonal wavelet bases, as defined above, with*

$$A\sum_{j,k}|c_{j,k}|^2 \le \left\| \sum_{j,k\in\mathbb{Z}} c_{j,k}\psi_{j,k}\right\|^2 \le B\sum_{j,k}|c_{j,k}|^2.$$

Then for all $u \in L^2(\mathbb{R})$,

$$A/B\sum_{j\in\mathbb{Z}}\|Q_j u\|^2 \le \|u\|^2 \le B/A\sum_{j\in\mathbb{Z}}\|Q_j u\|^2.$$

Proof. We can write $u = \sum_{j,k} c_{j,k}\psi_{j,k}$. It then follows that $Q_j u = \sum_k c_{j,k}\psi_{j,k}$, and

$$\begin{aligned}\|u\|^2 &= \left\|\sum_{j,k\in\mathbb{Z}} c_{j,k}\psi_{j,k}\right\|^2 \le B\sum_{j,k}|c_{j,k}|^2\\ &= B\sum_j\left(\sum_k |c_{j,k}|^2\right) \le B/A\sum_j\left\|\sum_k c_{j,k}\psi_{j,k}\right\|^2 \le B/A\sum_j \|Q_j\psi\|^2.\end{aligned}$$

The lower bound is proved analogously. □

A similar theorem holds, of course, for the splitting (3.5).

4. The biorthogonal splitting trick. A natural question is now whether wavelet packets can be generalized to the biorthogonal setting. Let us first generalize the "splitting trick."

LEMMA 4.1 (the "biorthogonal splitting trick"). *Suppose that the functions $f_k(x) = f(x-k)$ constitute a Riesz basis for their closed linear span E, with*

$$A|c_k|^2 \le \left\|\sum_k c_k f_k\right\|^2 \le B\sum_k |c_k|^2, \tag{4.1}$$

for all square integrable sequences $(c_k)_{k\in\mathbb{Z}}$. Define f^0, f^1 by

$$\hat{f}^\sigma(\xi) = m_\sigma(\xi/2)\hat{f}(\xi/2), \qquad \sigma = 0, 1,$$

with m_0 as above, and $m_1(\xi) = e^{-i\xi}\overline{\tilde{m}_0(\xi+\pi)}$. Then the functions $f_k^0 = \frac{1}{\sqrt{2}}f^0\left(\frac{x}{2}-k\right)$, $f_k^1(x) = \frac{1}{\sqrt{2}}f^1\left(\frac{x}{2}-k\right)$, $k\in\mathbb{Z}$ constitute a Riesz basis for E, with

$$A'\sum_k\left[|a_k|^2+|b_k|^2\right] \le \left\|\sum_k\left[a_k f_k^0 + b_k f_k^1\right]\right\|^2 \le B'\sum_k\left[|a_k|^2+|b_k|^2\right], \tag{4.2}$$

where

$$\begin{aligned}B' &= B(\max(M_0,\tilde{M}_0)+\Delta_0) + \frac{B-A}{2}(M_0\tilde{M}_0)^{1/2}\\ A' &= \left[A^{-1}(\max(M_0,\tilde{M}_0)+\Delta_0) + \frac{B-A}{2AB}(M_0\tilde{M}_0)^{1/2}\right]^{-1}\end{aligned}$$

and

$$M_0 = \sup_\xi \left[|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2\right]$$
$$\tilde{M}_0 = \sup_\xi \left[|\tilde{m}_0(\xi)|^2 + |\tilde{m}_0(\xi+\pi)|^2\right]$$
$$\Delta_0 = \sup_\xi |m_0(\xi)\tilde{m}_0(\xi+\pi) - m_0(\xi+\pi)\tilde{m}_0(\xi)|\,.$$

Remark. If the f_k are orthonormal to start with, and if $\tilde{m}_0 = m_0$ (orthonormal filter case), then $M_0 = \tilde{M}_0 = 1$, $\Delta_0 = 0$, $B = A = 1$, and the new bounds B', A' are also equal to 1. The estimates below can also be used to prove bounds of the type (4.2) where the constants B'', A'' are simply proportional to B and A respectively, namely

$$\begin{aligned} B'' &= B\left[\max(M_0, \tilde{M}_0) + (M_0\tilde{M}_0)^{1/2}\right] \\ A'' &= A\left[\max(M_0, \tilde{M}_0) + (M_0\tilde{M}_0)^{1/2}\right]^{-1}; \end{aligned} \tag{4.3}$$

these "simpler" bounds are less sharp, in the sense that they do not collapse to 1 if everything is orthonormal.

Proof.

1. For $(c_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, we denote by $c(\xi)$ the 2π-periodic function $c(\xi) = \sum_k c_k e^{-ik\xi} \in L^2([0, 2\pi])$. Then

$$\begin{aligned} \left\|\sum_k c_k f_k\right\|^2 &= \int_{-\infty}^{\infty} d\xi \left|\sum_k c_k e^{-ik\xi}\hat{f}(\xi)\right|^2 \\ &= \int_0^{2\pi} d\xi |c(\xi)|^2 \sum_k |\hat{f}(\xi+2\pi k)|^2, \end{aligned}$$

so that (4.1) is seen to be equivalent to $A/2\pi \le \sum_k |\hat{f}(\xi+2\pi k)|^2 \le B/2\pi$.

2. Define $\tilde{f} \in E$ by $\langle \tilde{f}, f_k\rangle = \delta_{0,k}$. Then the $\tilde{f}_\ell(x) = \tilde{f}(x-\ell)$, $\ell \in \mathbb{Z}$ constitute the dual Riesz basis for the f_k. In particular,

$$B^{-1}\sum_k |c_k|^2 \le \left\|\sum_k c_k \tilde{f}_k\right\|^2 \le A^{-1}\sum_k |c_k|^2 \tag{4.4}$$

and

$$\langle \tilde{f}_\ell, f_k\rangle = \delta_{k,\ell}.$$

(Both can easily be derived from $\hat{\tilde{f}}(\xi) = \frac{1}{2\pi}\hat{f}(\xi)[\sum_k |\hat{f}(\xi+2\pi k)|^2]^{-1}$.)

3. We start by proving that the f_k^0, f_ℓ^1 span all of E. Assume that $u \in E$ is orthogonal to all the f_ℓ^σ, $\ell \in \mathbb{Z}$, $\sigma = 0, 1$. Since the $\tilde{f}_k$ constitute a Riesz basis for E, we can write $u = \sum_k c_k \tilde{f}_k$. We have then

$$\begin{aligned} 0 = \langle u, f_\ell^\sigma\rangle &= \sqrt{2}\int_{-\infty}^{\infty} d\xi \overline{c(\xi)\hat{\tilde{f}}(\xi)}\hat{f}(\xi)m_\sigma(\xi)e^{-2i\ell\xi} \\ &= \sqrt{2}\int_0^{\pi} d\xi e^{-2i\ell\xi}\sum_{n\in\mathbb{Z}} m_\sigma(\xi+n\pi)\overline{c(\xi+n\pi)}\hat{f}(\xi+n\pi)\overline{\hat{\tilde{f}}(\xi+n\pi)}. \end{aligned}$$

Because m_σ and c are 2π-periodic, and $\sum_\ell \hat{f}(\xi+2\ell\pi)\overline{\hat{\tilde{f}}(\xi+2\ell\pi)} = \frac{1}{2\pi}$, this implies

$$m_\sigma(\xi)\overline{c(\xi)} + m_\sigma(\xi+\pi)\overline{c(\xi+\pi)} = 0 \quad \text{a.e.}$$

Multiplying with $\overline{\tilde{m}_\sigma(\xi)}$, and adding the two equations ($\sigma = 0,1$) leads to (again, almost everywhere)

$$\begin{aligned} 0 = \overline{c(\xi)}&\left[m_0(\xi)\overline{\tilde{m}_0(\xi)} + \overline{\tilde{m}_0(\xi+\pi)}m_0(\xi+\pi)\right] \\ &+ \overline{c(\xi+\pi)}\left[m_0(\xi+\pi)\overline{\tilde{m}_0(\xi)} - \overline{\tilde{m}_0(\xi)}m_0(\xi+\pi)\right] = \overline{c(\xi)}, \end{aligned}$$

which proves that $u = 0$.

4. Next we derive an upper bound on $\sum_k \left[a_k f_k^0 + b_k f_k^1\right]$. With the notation $F(\xi) = \sum_k |\hat{f}(\xi+2\pi k)|^2$, we have

$$\begin{aligned} \left\|\sum_k \left[a_k f_k^0 + b_k f_k^1\right]\right\|^2 &= 2\int_{-\infty}^{\infty} d\xi |\hat{f}(\xi)|^2 \left|\sum_k a_k e^{-2ik\xi} m_0(\xi) + \sum_k b_k e^{-2ik\xi} m_1(\xi)\right|^2 \\ \leq 2\int_0^\pi d\xi &\left[|a(2\xi)|^2 \left(F(\xi)|m_0(\xi)|^2 + F(\xi+\pi)|m_0(\xi+\pi)|^2\right)\right. \\ &+ |b(2\xi)|^2\left(F(\xi)|\tilde{m}_0(\xi+\pi)|^2 + F(\xi+\pi)|\tilde{m}_0(\xi)|^2\right) \\ &\left.+ 2|a(2\xi)||b(2\xi)||F(\xi)m_0(\xi)\tilde{m}_0(\xi+\pi) - F(\xi+\pi)m_0(\xi+\pi)\tilde{m}_0(\xi)|\right] \\ \leq 2\int_0^\pi d\xi &\left[|a(2\xi)|^2\left(\frac{B}{2\pi}M_0 + \frac{F(\xi)+F(\xi+\pi)}{2}|m_0(\xi)\tilde{m}_0(\xi+\pi) - m_0(\xi+\pi)\tilde{m}_0(\xi)|\right.\right. \\ &\left.+ \left|\frac{F(\xi)-F(\xi+\pi)}{2}\right|(M_0\tilde{M}_0)^{1/2}\right) \\ &+ |b(2\xi)|^2\left(\frac{B}{2\pi}\tilde{M}_0 + \frac{F(\xi)+F(\xi+\pi)}{2}|m_0(\xi)\tilde{m}_0(\xi+\pi) - m_0(\xi+\pi)\tilde{m}_0(\xi)|\right. \\ &\left.\left.+ \left|\frac{F(\xi)-F(\xi+\pi)}{2}\right|(M_0\tilde{M}_0)^{1/2}\right)\right] \\ \leq &\left[B(M_0+\Delta_0) + \frac{B-A}{2}(M_0\tilde{M}_0)^{1/2}\right]\sum |a_k|^2 \\ &+ \left[B(\tilde{M}_0+\Delta_0) + \frac{B-A}{2}(M_0\tilde{M}_0)^{1/2}\right]\sum |b_k|^2, \end{aligned}$$

which implies the upper bound in (4.2).

5. To derive the lower bound in (4.2), we introduce a dual family $\tilde{f}_k^0$, $\tilde{f}_k^1$, defined by $\tilde{f}_k^\sigma(x) = \frac{1}{\sqrt{2}}\tilde{f}^\sigma\left(\frac{x}{2}-k\right)$, with

$$\hat{\tilde{f}}^\sigma(\xi) = \tilde{m}_\sigma(\xi/2)\hat{\tilde{f}}(\xi/2),$$

where $\tilde{m}_0(\xi)$ is as above, and $\tilde{m}_1(\xi) = e^{-i\xi}\overline{m_0(\xi+\pi)}$. One easily proves $\langle f_\ell^\sigma, \tilde{f}_m^\tau\rangle = \delta_{\sigma,\tau}\delta_{\ell,m}$. For $\sigma = \tau = 0$, for instance,

$$\langle f_\ell^0, \tilde{f}_m^0\rangle = 2\int d\xi m_0(\xi)\overline{\tilde{m}_0(\xi)}\hat{f}(\xi)\overline{\hat{\tilde{f}}}(\xi)e^{2i(\ell-m)\xi}$$

$$
\begin{aligned}
&= \pi^{-1} \int_0^{2\pi} d\xi m_0(\xi) \overline{\tilde{m}_0(\xi)} e^{2i(\ell - m)\xi} \\
&= \pi^{-1} \int_0^{\pi} d\xi \left[m_0(\xi) \overline{\tilde{m}_0(\xi)} + m_0(\xi + \pi) \overline{\tilde{m}_0(\xi + \pi)} \right] e^{2i(\ell - m)\xi} \\
&= \delta_{\ell,m}.
\end{aligned}
$$

6. The same arguments as in point 4, together with (4.4), prove that

$$
\begin{aligned}
&\left\| \sum_k a_k \tilde{f}_k^0 + b_k \tilde{f}_k^1 \right\|^2 \\
&\quad \le \left[A^{-1} (\max(M_0, \tilde{M}_0) + \Delta_0) + \frac{A^{-1} - B^{-1}}{2} (M_0 \tilde{M}_0)^{1/2} \right] \sum_k \left[|a_k|^2 + |b_k|^2 \right] \\
&\quad = A'^{-1} \sum_k \left[|a_k|^2 + |b_k|^2 \right].
\end{aligned}
$$

The lower bound in (4.2) now follows from a simple duality argument.

$$
\begin{aligned}
\sum_k \left[|a_k|^2 + |b_k|^2 \right] &= \left\langle \sum_k \left[a_k f_k^0 + b_k f_k^1 \right], \sum_k \left[a_k \tilde{f}_k^0 + b_k \tilde{f}_k^1 \right] \right\rangle \\
&\le \left\| \sum_k \left[a_k f_k^0 + b_k f_k^1 \right] \right\| A'^{-1/2} \left(\sum_k \left[|a_k|^2 + |b_k|^2 \right] \right)^{1/2}. \qquad \square
\end{aligned}
$$

An immediate corollary is the following.

COROLLARY 4.2. *We assume the same as in Lemma* 4.1. *If* $u = u_0 + u_1$ *is the unique decomposition of* $u \in E$ *into* $u_\sigma = \sum_k c_k^\sigma f_k^\sigma$, *then*

$$
\frac{A'}{B'} \left[\|u_0\|^2 + \|u_1\|^2 \right] \le \|u\|^2 \le \frac{B'}{A'} \left[\|u_0\|^2 + \|u_1\|^2 \right].
$$

Proof. By the same argument as for Lemma 3.1. $\square$

5. Biorthogonal wavelet packets. We can now apply the biorthogonal splitting trick to the spaces V_0, W_j in the nonorthogonal decomposition (3.5). We start by choosing, among V_0 and all the W_j, an arbitrary subset of spaces to be split, and we apply the biorthogonal splitting trick to all of them. We end up with a different decomposition, in which all the "split" spaces are replaced by their two offspring. We can then repeat the procedure: choose an arbitrary subset, and split again. Every splitting corresponds to a replacement of the basis vectors as well. If, after L splitting steps, the subspace W_j has undergone $J \le L$ splittings, then the $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$, $k \in \mathbb{Z}$, will have been replaced by $\psi^J_{j;\epsilon_1,\ldots,\epsilon_k;\ell(x)} = 2^{(j-J)/2}\psi^J_{\epsilon_1,\ldots,\epsilon_J}(2^{j-J}x - \ell)$, $\epsilon_n = 0$ or 1, $\ell \in \mathbb{Z}$, with $\hat{\psi}^J_{\epsilon_1,\ldots,\epsilon_J}(\xi) = m_{\epsilon_1}(\xi/2), \ldots, m_{\epsilon_J}(2^{-J}\xi)\, \hat{\psi}(2^{-J}\xi)$. The following theorem tells us that as long as we confine ourselves to a finite number of splitting steps, the result is still a Riesz basis.

THEOREM 5.1. *Suppose we start from the decomposition* (3.5), *with*

$$
\begin{aligned}
(5.1)\qquad A\left[\sum_k |\alpha_k|^2 + \sum_{j,k} |\beta_{j,k}|^2\right] &\le \left\| \sum_{k\in\mathbb{Z}} \alpha_k \phi_{0,k} + \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}} \beta_{j,k}\psi_{j,k}\right\|^2 \\
&\le B\left[\sum_k |\alpha_k|^2 + \sum_{j,k} |\beta_{j,k}|^2\right].
\end{aligned}
$$

Let us denote by Ψ^L_λ the vectors obtained after L splitting steps, as described above (the label λ stands for $J, j, \epsilon_1, \ldots, \epsilon_K$ and ℓ). Then the Ψ^L_λ still constitute a Riesz basis, and

$$
(5.2)\qquad A_L \sum_\lambda |\gamma_\lambda|^2 \le \left\|\sum_\lambda \gamma_\lambda \Psi^L_\lambda\right\|^2 \le B_L \sum_\lambda |\gamma_\lambda|^2.
$$

The constants B_L, A_L are defined recursively by

$$
B_0 = B, A_0 = A, \beta_0 = B/A, \alpha_0 = A/B,
$$

and

$$
\begin{aligned}
(5.3)\qquad B_L &= \beta_{L-1}\left[B_{L-1}\mu_0 + (B_{L-1} - A_{L-1})\,\nu_0\right] \\
A_L &= \alpha_{L-1}\left[A^{-1}_{L-1}\mu_0 + \left(A^{-1}_{L-1} - B^{-1}_{L-1}\right)\nu_0\right]^{-1}
\end{aligned}
$$

with

$$
\begin{aligned}
(5.4)\qquad \beta_L &= \beta_{L-1}\left[B_{L-1}\mu_0 + (B_{L-1} - A_{L-1})\,\nu_0\right]\left[A^{-1}_{L-1}\mu_0 + \left(A^{-1}_{L-1} - B^{-1}_{L-1}\right)\nu_0\right] \\
\alpha_L &= \alpha_{L-1}\left[A^{-1}_{L-1}\mu_0 + \left(A^{-1}_{L-1} - B^{-1}_{L-1}\right)\nu_0\right]^{-1}\left[B_{L-1}\mu_0 + (B_{L-1} - A_{L-1})\,\nu_0\right]^{-1}
\end{aligned}
$$

and

$$
\begin{aligned}
\mu_0 &= \max(M_0, \tilde{M}_0) + \Delta_0 \\
\nu_0 &= \tfrac{1}{2}(M_0\tilde{M}_0)^{1/2}
\end{aligned}
$$

($M_0, \tilde{M}_0, \Delta_0$ as defined in Lemma 4.1*).*

Proof.

1. We will work by induction on L, the number of splittings. Suppose that we have gone through ℓ splitting steps, resulting in a (nonorthogonal) decomposition of $L^2(\mathbb{R})$, i.e.,

$$
L^2(\mathbb{R}) = \bigoplus_m E_{\ell,m};
$$

in each $E_{\ell,m}$ we have a Riesz basis $F_{\ell,m;\,k}$, $k \in \mathbb{Z}$. Assume that

$$
A_\ell \sum_{m,k} |c_{m,k}|^2 \le \left\|\sum_{m,k} c_{m,k} F_{\ell,m;\,k}\right\|^2 \le B_\ell \sum_{m,k} |c_{m,k}|^2
$$

and that for arbitrary $u \in L^2(\mathbb{R})$, $u = \sum_m u_{\ell,m}$, with $u_{\ell,m} \in E_{\ell,m}$, we have

$$\alpha_\ell \sum_m \|u_{\ell,m}\|^2 \le \|u\|^2 \le \beta_\ell \sum_m \|u_{\ell,m}\|^2. \tag{5.5}$$

We now choose an arbitrary subset of the $\{E_{\ell,m}; m \in \mathbb{Z}\}$ of spaces to be split, and we apply the biorthogonal splitting trick to each of them. If $E_{\ell,n}$ is a space that gets split into $E^0_{\ell,n} \oplus E^1_{\ell,n}$, then for arbitrary

$$u_{\ell,n} = \sum_k c^0_{n,k} F^0_{\ell,n;k} + \sum_k c^1_{n,k} F^1_{\ell,n;k} = u^0_{\ell,n} + u^1_{\ell,n},$$

applying Lemma 4.1 leads to

$$\begin{aligned} &\left[A_\ell^{-1}\mu_0 + \left(A_\ell^{-1} - B_\ell^{-1}\right)\nu_0\right]^{-1} \sum_{\sigma=0,1}\sum_k |c^\sigma_{n,k}|^2 \\ &\quad \le \|u_{\ell,n}\|^2 \le \left[B_\ell\mu_0 + (B_\ell - A_\ell)\,\nu_0\right] \sum_{\sigma=0,1}\sum_k |c^\sigma_{n,k}|^2. \end{aligned}$$

Bringing all these inequalities together, combining with (5.5), and relabeling the $F^\sigma_{\ell,n;\,k}$ as $F_{\ell+1,m;\,k'}$, we obtain

$$\begin{aligned} &\alpha_\ell \left[A_\ell^{-1}\mu_0 + \left(A_\ell^{-1} - B_\ell^{-1}\right)\nu_0\right]^{-1} \sum_k |c_{m,k}|^2 \le \left\|\sum_{m,k} c_{m,k} F_{\ell+1,m;k}\right\|^2 \\ &\quad \le \beta_\ell \left[B_\ell\mu_0 + (B_\ell - A_\ell)\,\nu_0\right] \sum_k |c_{m,k}|^2. \end{aligned} \tag{5.6}$$

2. From Corollary 4.2 and (5.5) we also obtain

$$\alpha_{\ell+1} \sum_m \|u_{\ell+1,m}\|^2 \le \|u\|^2 \le \beta_{\ell+1} \sum_m \|u_{\ell+1,m}\|^2, \tag{5.7}$$

with

$$\begin{aligned} \beta_{\ell+1} &= \beta_\ell \left[B_\ell\mu_0 + (B_\ell - A_\ell)\,\nu_0\right]\left[A_\ell^{-1}\mu_0 + \left(A_\ell^{-1} - B_\ell^{-1}\right)\nu_0\right], \\ \alpha_{\ell+1} &= \alpha_\ell \left[B_\ell\mu_0 + (B_\ell - A_\ell)\,\nu_0\right]^{-1}\left[A_\ell^{-1}\mu_0 + \left(A_\ell^{-1} - B_\ell^{-1}\right)\nu_0\right]^{-1}. \end{aligned}$$

(5.6) and (5.7) can be used for the next induction step.

3. To start it all (at $L = 0$), we need (5.1), together with Lemma 3.1, which leads to

$$\begin{aligned} A_0 &= A, & B_0 &= B \\ \alpha_0 &= A/B, & B_0 &= B/A. \end{aligned} \qquad \square$$

Remarks.

1. If $A = B = 1$ and $m_0 = \tilde{m}_0$, then $M_0 = \tilde{M}_0 = 1$, $\Delta_0 = 0$, and $A_L = B_L = 1$ for every L; in the orthonormal case we recover the exact estimates for orthonormal wavelet packets.

2. In the nonorthonormal case, B_L and A_L^{-1} increase very rapidly with L: one easily checks that (5.2) and (5.3) imply that $B_L \sim CM^{2^L}$ for large L. In Chui and Li [2] a different technique is used to derive bounds similar to (5.2), with $\log B_L \sim$

const. L, i.e., only exponential growth for B_L. The bounds of Chui and Li do not seem to collapse to the optimal bounds 1 in the orthonormal case, however. They also restrict themselves to the case where one chooses, at every splitting step, to split every available subspace; this is probably not a crucial restriction. The estimates in the next section show that B_L grows at least exponentially with L.

3. For every possible choice of splittings, the dual basis of the resulting Riesz basis can be constructed by applying exactly the same choice of splittings on the original dual decomposition (with the $\tilde{\psi}_{j,k}$ instead of the $\psi_{j,k}$), and using $\tilde{m}_0, \tilde{m}_1$ instead of m_0, m_1 at every splitting step. This follows from the constructions in §4; see also Chui and Li [2].

6. Instability of arbitrary biorthogonal wavelet packets. If we choose to restrict to at most L splittings, then the previous section tells us that we will still have a Riesz basis, even though the constants involved may be large. In the orthonormal case, a very beautiful special wavelet packet basis resulted from splitting every W_j exactly j times. In this decomposition, the total number of splitting steps is not limited, and Theorem 5.1 does not guarantee that the biorthogonal analog leads to a Riesz basis. We shall show in this section that in fact we don't have a Riesz basis (except in the orthonormal case). Define, as in the orthonormal case,

$$\hat{\psi}_\epsilon(\xi) = \left[\prod_{j=1}^{N} m_{\epsilon_j}(2^{-j}\xi)\right] \hat{\phi}(2^{-N}\xi),$$

where $\epsilon = (\epsilon_1, \dots, \epsilon_N)$ is a sequence of length N ($|\epsilon| = N$) consisting of only zeros and ones. We start by proving several lemmas about the $\hat{\psi}_\epsilon$.

LEMMA 6.1. *There exists a constant $C > 0$ such that*

$$\int d\xi |\hat{\psi}_\epsilon(\xi)|^2 \geq C \int_{|\xi| \leq 2^N \pi} d\xi \prod_{j=1}^{N} |m_{\epsilon_j}(2^{-j}\xi)|^2 \tag{6.1}$$

for all N and ϵ with $|\epsilon| = N$.

Proof.

1. Remember (see §3) that there exists a compact set K, congruent with $[-\pi, \pi]$ modulo 2π, so that $|\hat{\phi}(\xi)| \geq C^2 > 0$ for all $\xi \in K$.

2. We have

$$|\hat{\psi}_\epsilon(\xi)| = \prod_{j=1}^{N} |m_{\epsilon_j}(2^{-j}\xi)| |\hat{\phi}(2^{-N}\xi)|,$$

hence

$$\begin{aligned}
\int d\xi |\hat{\psi}_\epsilon(\xi)|^2 &\geq \int_{\xi \in 2^N K} d\xi \prod_{j=1}^{N} |m_{\epsilon_j}(2^{-j}\xi)|^2 |\hat{\phi}(2^{-N}\xi)|^2 \\
&\geq C \int_{\xi \in 2^N K} d\xi \prod_{j=1}^{N} |m_{\epsilon_j}(2^{-j}\xi)|^2 \\
&= C \int_{|\xi| \leq 2^N \pi} d\xi \prod_{j=1}^{N} |m_{\epsilon_j}(2^{-j}\xi)|^2,
\end{aligned}$$

where we have used the congruency of K with $[-\pi, \pi]$, and the 2π-periodicity of the m_{ϵ_j}. □

LEMMA 6.2. *Define* $p(\xi) = |m_0(\xi)|^2 + |m_1(\xi)|^2$. *Then for some* $C > 0$,

$$\sum_{\epsilon \in S_N} \|\psi_\epsilon\|^2 \geq C 2^N \int_{|\xi| \leq \pi} d\xi \prod_{j=0}^{N-1} p(2^j \xi), \tag{6.2}$$

where S_N *is the set of all sequences* $\epsilon = (\epsilon_1, \dots, \epsilon_N)$ *of length* N *and consisting of only zeros and ones.*

Proof. The proof follows immediately from (6.1) by summing over the 2^N sequences ϵ with length N, and by changing the integration variable. □

The following lemma will allow us to compute a lower bound for the right-hand side of (6.2).

LEMMA 6.3. *The function* $p(\xi) = |m_0(\xi)|^2 + |m_1(\xi)|^2$ *satisfies*

$$p(\xi) p(\xi + \pi) \geq 1.$$

Moreover, if $m_0 \neq \tilde{m}_0$ *(nonorthogonal case), then*

$$p(\xi) p(\xi + \pi) > 1 \quad a.e.$$

Proof.

1. We know (see §4) that

$$m_0(\xi) \overline{\tilde{m}_0(\xi)} + m_0(\xi + \pi) \overline{\tilde{m}_0(\xi + \pi)} = 1. \tag{6.3}$$

By Cauchy–Schwarz, this implies

$$\left[|m_0(\xi)|^2 + |\tilde{m}_0(\xi + \pi)|^2 \right] \left[|\tilde{m}_0(\xi)|^2 + |m_0(\xi + \pi)|^2 \right] \geq 1 \tag{6.4}$$

or

$$p(\xi) p(\xi + \pi) \geq 1 \left(\text{use } m_\sigma(\xi) = e^{-i\xi} \overline{\tilde{m}_{1-\sigma}(\xi + \pi)} \right).$$

2. Equality in (6.4) is only possible for those ξ for which

$$\tilde{m}_0(\xi) = \alpha(\xi) m_0(\xi), \qquad \overline{m_0(\xi + \pi)} = \alpha(\xi) \overline{\tilde{m}_0(\xi + \pi)}$$

for some $\alpha(\xi)$. For such ξ,

$$\tilde{m}_0(\xi) \overline{\tilde{m}_0(\xi + \pi)} - \overline{m_0(\xi + \pi)} m_0(\xi) = 0. \tag{6.5}$$

3. Suppose that (6.5) were true for all ξ. If we extend the trigonometric polynomials from $z = e^{-i\xi}$ on the torus to all $z \in \mathbb{C}$ (extending $M_0(e^{-i\xi}) = m_0(\xi)$), then the identity (6.5) would still hold for all z,

$$\tilde{M}_0(z) \overline{\tilde{M}}(-z^{-1}) - M_0(z) \overline{M}_0(-z^{-1}) = 0, \tag{6.6}$$

where for $A(z) = \sum_n a_n z^n$, we use the notation $\bar{A}(z) = \sum_n \bar{a}_n z^n$. On the other hand, extension of (6.3) gives

$$M_0(z) \overline{\tilde{M}}_0(z^{-1}) + M_0(-z) \overline{\tilde{M}}_0(-z^{-1}) = 1,$$

which means that $M_0(z)$ and $\overline{\tilde{M}}(-z^{-1})$ share no zeros. It then follows from (6.6) that $\tilde{M}_0(z)$ is zero whenever $M_0(z)$ is. Similarly one concludes that $M_0(z)$ is zero whenever $\tilde{M}_0(z)$ is. Since both are polynomials (up to multiplication by an integer power of z), $M_0 \equiv \tilde{M}_0$ follows.

4. If $\tilde{m}_0 \neq m_0$, one finds therefore that the left-hand side of (6.5) is a nontrivial trigonometric polynomial. It follows that (6.5) can only hold in a finite number of ξ_j. Consequently, (6.4) is a strict inequality except in this finite number of ξ_j, which implies that $p(\xi)p(\xi+\pi) > 1$ almost everywhere. □

We are now ready for the following theorem.

THEOREM 6.4. *There exist $C > 0$, $\lambda > 1$ so that, for all $N \in \mathbb{N}$, $N \geq 1$,*

$$\sum_{\substack{\epsilon \\ |\epsilon|=N}} \|\psi_\epsilon\|^2 \geq C 2^N \lambda^N. \tag{6.7}$$

Proof.

1. By Lemma 6.2, we only need to prove that

$$\int_{-\pi}^{\pi} d\xi \prod_{j=0}^{N-1} p(2^j \xi) \geq C\lambda^N.$$

2. By Jensen's inequality,

$$\log\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} d\xi \prod_{j=0}^{N-1} p(2^j\xi)\right] \geq \frac{1}{2\pi}\int_{-\pi}^{\pi} d\xi \log\left[\prod_{j=0}^{N-1} p(2^j\xi)\right]$$
$$= \frac{1}{2\pi}\sum_{j=0}^{N-1}\int_{-\pi}^{\pi} d\xi \log p(2^j\xi) = \frac{N}{2\pi}\int_{-\pi}^{\pi} d\xi \log p(\xi).$$

3. Since $p(\xi)p(\xi+\pi) > 1$ a.e., it follows that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} d\xi \log p(\xi) = \frac{1}{2\pi}\int_{0}^{\pi} d\xi \log[p(\xi)p(\xi+\pi)] = \gamma > 0. \qquad \Box$$

Consequently,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} d\xi \prod_{j=0}^{N-1} p(2^j\xi) \geq e^{N\gamma}.$$

Remark. The argument in this proof is borrowed from the proof of Theorem 3 in Coifman, Meyer, and Wickerhauser [10].

This suffices to prove the instability claimed above. If the collection

$$\mathcal{C} = \{\phi(\cdot - k); k \in \mathbb{Z}\} \cup \{\psi_\epsilon(\cdot - k), k \in \mathbb{Z} \text{ and } |\epsilon| = N, \epsilon_N = 1, N \in \mathbb{N}\backslash\{0\}\}$$

were a Riesz basis for $L^2(\mathbb{R})$, then it would follow that the L^2-norms of all these functions could be bounded uniformly by some constant C. (A Riesz basis is the image of an orthonormal basis under a continuous map.) In particular it would follow that, for all $N \in \mathbb{N}$,

$$\sum_{\epsilon \in S_N} \|\psi_\epsilon\|^2 \leq C\#S_N = C2^N.$$

This is contradicted by (6.7); the collection $\mathcal{C}$ does therefore not constitute a Riesz basis.

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