Wiener Measures for Quantum Mechanical Path Integrals

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Our purpose here is to show that it is possible to represent certain quantum mechanical evolution operators by path integrals with mathematically well-defined measures. For the Hamiltonians we consider here, this measure will be a Wiener measure on phase space. To achieve this goal we exploit the overcompleteness of the coherent states.

Coherent states are defined as

$$|z\rangle \equiv |p,q\rangle \equiv e^{i(pQ-qP)}|0\rangle$$
$$\equiv e^{-\frac{1}{2}|z|^2} e^{z\,A^\dagger}\,|0\rangle \quad ,$$

where $|0\rangle$ is the harmonic oscillator ground state, and where $z \equiv (q+ip)/\sqrt{2}$.
In the Schrödinger representation one has

$$\langle x|z\rangle = \pi^{-\frac{1}{4}} \exp\,[ip(x-q/2) - (x-q)^2/2]. \tag{1}$$

The overcompleteness of the coherent states is illustrated by the overlap function

$$\langle z|z'\rangle = \exp[-\tfrac{1}{2}\,|z|^2 - \tfrac{1}{2}\,|z'|^2 + z^*z']$$

which shows that these states are never mutually orthogonal. Nevertheless, one of the most useful properties of the coherent states is the "resolution of the identity"

$$I = \int |z\rangle\langle z|\;d\mu \tag{2}$$

where $d\mu \equiv \frac{1}{2\pi}\,dp\,dq$, integrated over $\mathbb{R}^2$.

Equation (2) can be used to represent operators by means of their matrix elements between coherent states. If B is a (bounded) operator, we can always write B as

$$B = \int |z''\rangle\langle z''|B|z'\rangle\langle z'|\;\;d\mu''d\mu' \quad . \tag{3}$$

Due to the overcompleteness of the coherent states, many other functions can be found to replace $\langle z''|B|z'\rangle$ without invalidating (3). These functions form an equivalence class (labelled by B), a generic element of which we denote by $\langle z''|B|z'\rangle_{E.C.}$, and therefore

$$B = \int |z''\rangle\langle z''|B|z'\rangle_{E.C.}\;\langle z'|\;d\mu''d\mu' \quad . \tag{4}$$

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The class of dynamical systems we shall consider here is given by the Hamiltonians

$$H(t) = \frac{1}{2}(P^2 + Q^2 - 1) + s(t)Q \quad .$$

The coherent state matrix element of the evolution operator can be written as a formal coherent state path integral[1],

$$\begin{aligned}\langle z'',t''|z',t'\rangle &\equiv \langle z''|T\exp[-i\int H(t)dt]|z'\rangle \\ &= \mathcal{N}\int \exp(i\int\{\tfrac{1}{2}[p(t)\dot{q}(t)-q(t)\dot{p}(t)] \\ &\quad -\tfrac{1}{2}[p^2(t)+q^2(t)]-s(t)q(t)\}dt)\,\Pi\, dp(t)dq(t) \quad . \end{aligned} \qquad (5)$$

The "volume element" $\Pi\, dp(t)dq(t)$ does not define a genuine measure, and the right hand side of (5) should be understood as a limit[1],

$$\begin{aligned}\langle z'',t''|z',t'\rangle &= \lim_{\varepsilon\to 0}\int\ldots\int \exp(\textstyle\sum\{\tfrac{i}{2}[p_k(q_{k+1}-q_k)-q_k(p_{k+1}-p_k)] \\ &\quad -\tfrac{1}{4}[(p_{k+1}-p_k)^2+(q_{k+1}-q_k)^2]-\tfrac{i}{2}\varepsilon(q_{k+1}-ip_{k+1})(q_k+ip_k) \\ &\quad -\tfrac{i}{2}\varepsilon\, s_k[q_{k+1}+q_k+i(p_{k+1}-p_k)]\})\,\Pi(dp_k dq_k/2\pi)\end{aligned}$$

We claim that it is however possible to write a path integral over a genuine measure for a member in the same equivalence class as $\langle z'',t''|z',t'\rangle$ in the fashion

$$\begin{aligned}\langle z'',t''|z',t'\rangle_{E.C.} &= 2\pi\int \exp(\int\{(\tfrac{i}{2}+\tfrac{1}{\nu})[p(t)\dot{q}(t)-q(t)\dot{p}(t)] \\ &\quad -\tfrac{1}{2}(i+\tfrac{1}{\nu})[p^2(t)+q^2(t)]-\tfrac{1}{2\nu}[s^2(t)-\nu^2] \\ &\quad -s(t)[(i+\tfrac{1}{\nu})q(t)+\tfrac{1}{\nu}\dot{p}(t)]\}dt)\,d\rho_W(p)\; d\rho_W(q) \quad , \end{aligned} \qquad (6)$$

where each ρ_W is a Wiener measure, and ν can be arbitrarily chosen in $\mathbb{R}_+^*$ (every $\nu>0$ leads to a member of the equivalence class). For p the measure is pinned at the initial point p' and at the final point p", and is normalised such that

$$\int d\rho_W(p) = [2\pi\nu(t''-t')]^{-\frac{1}{2}}\exp\{-(p''-p')^2/[2\nu(t''-t')]\} \quad ;$$

the Wiener measure for q is defined entirely analogously. The integrand in (6) is a well-defined stochastic integral. For example, the term $\int(p\dot{q}-q\dot{p})dt$ should be understood as $\int pdq-\int qdp$; since p and q are independent stochastic variables all procedures to define these integrals are equivalent. The construction[2] of (6) exploits well-known properties of the harmonic oscillator eigenstates and of the Weyl operators. Note that while there are similarities between the integrands in (5) and (6), some coefficients are different: we have a coefficient $(\frac{i}{2}+\frac{1}{\nu})$ for $(p\dot{q}-q\dot{p})$ in (6) instead of $\frac{i}{2}$ in (5), and $-\frac{1}{2}(i+\frac{1}{\nu})$ for (p^2+q^2) instead of $-\frac{i}{2}$. Moreover an extra term $-\frac{1}{2\nu}\int dt[s^2(t)-\nu^2]$ appears in (6) (we assume s to be square integrable).

Some further remarks regarding this construction are:

1. One can incorporate the term $-\frac{1}{2\nu}(p^2+q^2)$ in the integrand in (6) into the Wiener measures, and so write $\langle z'',t''|z',t'\rangle$ as an integral over an associated Ornstein-Uhlenbeck measure.
2. For $t''-t' \to \infty$, with ν fixed, the equivalence class function $\langle z'',t''|z',t'\rangle_{E.C.}$ converges to the matrix element $\langle z'',t''|z',t'\rangle$; this happens whatever the fixed value is assigned to ν . Hence (6) gives a genuine path integral expression for the matrix element $\langle z'',t''|z',t'\rangle$ in the limit when the time interval diverges.
3. Alternatively in the limit $\nu \to \infty$, with $t''-t'$ fixed, the equivalence class function $\langle z'',t''|z',t'\rangle_{E.C.}$ converges to the matrix element $\langle z'',t''|z',t'\rangle$. This result may be heuristically seen since (6) formally converges to (5) in the limit $\nu \to \infty$.
4. One can use (1) and (4) to write the matrix element $\langle x'',t''|x',t'\rangle$ as a well-defined path integral,

$$\langle x'',t''|x',t'\rangle = \int \langle x''|z''\rangle \langle z'',t''|z',t'\rangle_{E.C.} \langle z'|x'\rangle d\mu'' d\mu' \ .$$

   Carrying out the integral over p, $\langle x'',t''|x',t'\rangle$ is then expressed as a path integral over a Wiener measure in q alone; the integrand, though very complicated, is still a well-defined stochastic integral.
5. An analogous construction holds for other kinematical systems as well. In particular, for spin systems, special spin coherent state matrix elements in the equivalence class of the evolution operator for certain systems may be expressed as a path integral involving Wiener measure on the surface of the unit sphere.[3]

References

1. See e.g., J. R. Klauder, Acta Physica Austriaca, Suppl. XXII, 3 (1980); in Path Integrals, ed. by G. J. Papadopoulos and J. T. Devreese (Plenum, 1978), p. 5.
2. I. Daubechies and J. R. Klauder, "Constructing Measures for Path Integrals", to be published in J. Math. Phys.
3. J. R. Klauder, "Constructing Measures for Spin-Variable Path Integrals", submitted for publication.