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A connection between propositional systems in Hilbert spaces and von Neumann algebras

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Abstract. A theorem of Bade proves that for a complete Boolean sublattice $\mathscr{L}$ of $\mathscr{P}(\mathscr{H})$ the following holds:

$\mathscr{L} = \{P \in \mathscr{L}''; P$ is orthogonal projection operator$\}$

We prove that this theorem does not hold for the physically interesting class of non-Boolean propositional systems embedded in a $\mathscr{P}(\mathscr{H})$; we derive however a necessary and sufficient condition under which the theorem does hold. This condition is automatically satisfied if the propositional system is Boolean.

1. Introduction

There exist several formalisms for the description of quantum phenomena. One of these is the axiomatic approach of Jauch and Piron [1] where one starts from some intuitive and physically very comprehensive ideas. Another one is the algebraic approach. It is amusing to note that they were both initiated by von Neumann [2], [3]. The well known Hilbert space approach can be considered as a special case of the axiomatic approach [4] as well as of the algebraic approach.

There is however a sensible difference between these two formalisms: the axiomatic approach, as said before, has a more solid physical foundation but it is not very practical for numerical calculations while the algebraic approach, although physically less justified, is mathematically much more developed and is a better tool for practical calculations.

It seems therefore interesting to establish a connection between the physically more satisfying axiomatic lattice approach and the mathematically more convenient algebraic approach. A first step in this direction was made by Bade [5], who proved that a simple connection exists between complete distributive lattices of projections in a complex Hilbert space and commutative von Neumann algebras. We prove here an analogous theorem for a class of non-distributive lattices, namely the physically interesting propositional systems of the axiomatic approach [1].

In Section 2 we recall some definitions and basic theorems, including Bade's theorem, and we formulate the problem.

In Section 3 we state and prove our main theorem and give some counterexamples. In Section 4 we prove a generalization of this theorem.

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[1]) Wetenschappelijke medewerkers bij het Interuniversitair Instituut voor Kernwetenschappen (in het kader van navorsingsprogramma 21 EN).

2. Definitions and basic notions – formulation of the problem

According to the axiomatic approach the structure of the set of propositions corresponding to 'yes–no' experiments on a physical system is that of a complete, orthocomplemented, weakly modular and atomic lattice which satisfies the covering law. Such a lattice is called a propositional system. If the physical system has no superselection rules, the propositional system is irreducible. For the definition of a propositional system and of irreducibility we refer the reader to [6]; for more details he can consult [1].

An important example of such an irreducible propositional system is given by the following construction: let $\mathcal{H}$ be a complex Hilbert space, $\mathcal{P}(\mathcal{H})$ the collection of all the closed subspaces of $\mathcal{H}$.

If F, G are two elements of $\mathcal{P}(\mathcal{H})$, we say that $G < F$ iff $G \subset F$ set-theoretically. This defines a partial order relation on $\mathcal{P}(\mathcal{H})$. For $(G_i)_{i\in I}$ in $\mathcal{P}(\mathcal{H})$ we define

$$\bigwedge_{i\in I} G_i = \bigcap_{i\in I} G_i \quad \text{(set-theoretical intersection)} \tag{2.1}$$

$$\bigvee_{i\in I} G_i = \overline{\text{Span } G_i} \quad \text{(closure of the subspace generated by the } G_i\text{)} \tag{2.2}$$

For G in $\mathcal{P}(\mathcal{H})$ we define

$$G' = G^{\perp} \tag{2.3}$$

where $G^{\perp}$ is the closed subspace orthogonal to G. It is easy to check that $\mathcal{P}(\mathcal{H})$, equipped with this structure, is indeed an irreducible propositional system.

Remark. With every closed subspace G of $\mathcal{H}$ corresponds a unique orthogonal projection, namely the projection with image G. On the other hand the image $P\mathcal{H}$ of any orthogonal projection P is a closed subspace of $\mathcal{H}$. We will often use this correspondence to identify the closed subspaces of $\mathcal{H}$ with the orthogonal projections in $\mathcal{L}(\mathcal{H})$; we will even use the same symbol $\mathcal{P}(\mathcal{H})$ for both, and depending on the context an element A of $\mathcal{P}(\mathcal{H})$ may either be a closed subspace of $\mathcal{H}$ or an orthogonal projection in $\mathcal{L}(\mathcal{H})$. If for instance x is a non-zero vector in $\mathcal{H}$, the symbol $\bar{x}$ will be used to denote the subspace $\mathbf{C}x$ as well as the orthogonal projection on this subspace. This identification will never lead to confusion.

A propositional system $\mathcal{L}$ will be called a propositional subsystem of $\mathcal{P}(\mathcal{H})$ iff $\mathcal{L}$ is a sublattice of $\mathcal{P}(\mathcal{H})$ (i.e. $\mathcal{L} \subset \mathcal{P}(\mathcal{H})$ and $\vee$, $\wedge$ on $\mathcal{L}$ are the restrictions to $\mathcal{L}$ of the operations $\vee$, $\wedge$ defined on $\mathcal{P}(\mathcal{H})$) with the additional property

$$P' = P^{\perp} \wedge I \tag{2.4}$$

where I is the maximal element of $\mathcal{L}$.

Remark. Some concepts are much easier to define for a propositional subsystem $\mathcal{L}$ of a $\mathcal{P}(\mathcal{H})$ than for a general propositional system. For example:

– in a general propositional system $\mathcal{L}$ two elements are said to be compatible iff

$$(P \wedge Q') \vee Q = P \wedge Q;$$

for a propositional subsystem $\mathcal{L}$ of $\mathcal{P}(\mathcal{H})$, this is equivalent with the statement that the projections P and Q commute.

– in a general propositional system $\mathscr{L}$, the center Z is defined to be the set of all elements of $\mathscr{L}$ which are compatible with any elements of $\mathscr{L}$; for a propositional subsystem $\mathscr{L}$ of $\mathscr{P}(\mathscr{H})$, the center Z is given by

$$Z=\{P\in\mathscr{L};\forall Q\in\mathscr{L}:[P,Q]=0\}.$$

One remarkable property of irreducible propositional systems is their showing a structure which resembles that of $\mathscr{P}(\mathscr{H})$ in the example constructed above. This is proven by C. Piron's representation theorem (see [1], [4]) which states that any irreducible propositional system with rank at least 4 (i.e. with at least four orthogonal atoms) is isomorphic to the lattice of all biorthogonal subspaces of a vectorspace V over some field $\mathbf{K}$, the orthocomplementation defining an involutive anti-automorphism on $\mathbf{K}$ and a non degenerate sesquilinear form on V. One can even prove (see [1]) that this vectorspace V turns out to be a Hilbertspace if we take the field $\mathbf{K}$ to be $\mathbf{C}$ and the involutive anti-automorphism of $\mathbf{C}$ to be the usual conjugation, which are two restrictions one usually makes when studying physics. Since the set of all biorthogonal manifolds of a Hilbertspace is exactly $\mathscr{P}(\mathscr{H})$, we are then reduced to the case considered in the example.

In the algebraic approach one takes a von Neumann algebra of a C^*-algebra to represent the observables of the physical system. In the cases where a C^*-algebra is used, one often studies representations of the C^*-algebra, which amounts again to study von Neumann algebras.

There exists a definition for an abstract von Neumann algebra (i.e. a definition independent of a Hilbertspace $\mathscr{H}$), but one usually studies concrete von Neumann algebras, i.e. subalgebras of a $\mathscr{L}(\mathscr{H})$ where $\mathscr{H}$ is some complex Hilbertspace. One of the possible definitions of a von Neumann algebra is the following (see [8]).

Definition. $m\subset\mathscr{L}(\mathscr{H})$ is called a von Neumann algebra if m is an involutive subalgebra of $\mathscr{L}(\mathscr{H})$ such that $m=m''$ (i.e. m is equal to its bicommutant).

Using this definition, one can prove easily the following theorem.

2.1 Theorem. *Let $\mathscr{H}$ be a complex Hilbertspace, m a von Neumann algebra in $\mathscr{H}$. Then $P(m)=\{P\in m; P$ is an orthogonal projection$\}$ is a complete, orthocomplemented, weakly modular lattice with respect to the in* (2.1)–(2.3) *defined operations.*

This lattice is irreducible iff m is a factor; it is irreducible and atomic iff m is a factor of type I. In the latter case $P(m)$ satisfies automatically the covering law, which implies that $P(m)$ is an irreducible propositional system. (For two simple definitions of a factor and a factor of type I, see [9]).

Proof. Let $(P_i)_{i\in I}$ be a family of projections in m. For any projection Q in m' we have $[P_i,Q]=0$ $\forall i\in I$. This implies that for any $i\in I$ we can write P_i as

$$P_i=P_{i,1}+P_{i,2}=P_{i,1}\vee P_{i,2}$$

where

$$P_{i,1}=QP_i<Q \quad\text{and}\quad P_{i,2}<Q'$$

Hence

$$\bigwedge_{i\in I} P_i = \bigwedge_{i\in I} (P_{i,1}+P_{i,2})$$

$$= \Big(\bigwedge_{i\in I} P_{i,1}\Big) \vee \Big(\bigwedge_{i\in I} P_{i,2}\Big) \qquad (\forall i, j : [P_{i,1}, P_{j,2}]=0)$$

Since now

$$\bigwedge_{i\in I} P_{i,1} < Q, \qquad \bigwedge_{i\in I} P_{i,2} < Q', \quad \text{this implies that} \quad \Big[\bigwedge_{i\in I} P_i, Q\Big]=0$$

This is true for any projection Q in m', which implies

$$\bigwedge_{i\in I} P_i \in m'' = m \tag{2.5}$$

Analogously one proves that

$$\bigvee_{i\in I} P_i \in m \tag{2.6}$$

(2.5) and (2.6) together prove that $P(m)$ is a complete lattice.

For $P \in P(m)$, we have immediately $P' = \mathbf{1} - P \in m$, which implies that $P(m)$ is a complete orthocomplemented lattice. Weak modularity states that if P is smaller than Q, P and Q are compatible. This is trivially the case here.

m is a factor iff the only projections in m which commute with $P(m)$ are 0 and $\mathbf{1}$. This is equivalent to saying that the center of the lattice $P(m)$ is trivial, hence to saying that $P(m)$ is irreducible.

m is a factor of type I iff m is a factor such that any projection P in m contains a minimal projection. This is equivalent to saying that $P(m)$ is irreducible and atomic. In this last case one can check immediately that the covering law holds.

So once a von Neumann algebra is given, we know that the set of its projections is a complete, orthocomplemented, weakly modular lattice generating the von Neumann algebra, i.e.

$$P(m)'' = m$$

One can now wonder if the converse is true. If one has a complete, orthocomplemented, weakly modular sublattice $\mathscr{L}$ of $\mathscr{P}(\mathscr{H})$, (with the operations defined by (2.1), (2.2) and (2.4)), then $\mathscr{L}$ generates a von Neumann algebra, namely $\mathscr{L}''$ ($\mathscr{L}$ is self-adjoint, see [8] p. 2). It is trivial to check that $\mathscr{L} \subset P(\mathscr{L}'')$. One can ask oneself whether the other inclusion holds as well, i.e. whether

$$\mathscr{L} = P(\mathscr{L}'') \tag{2.7}$$

If the lattice $\mathscr{L}$ does not contain $\mathbf{1}_{\mathscr{H}}$, one immediately sees that

$$\mathbf{1}_{\mathscr{H}} \in P(\mathscr{L}'') \setminus \mathscr{L}$$

One can however always restrict oneself to the case where $\mathbf{1}_{\mathscr{H}} \in \mathscr{L}$. Indeed, we can prove the following theorem:

2.2 Theorem. *Let $\mathscr{H}$ be a complex Hilbertspace. Let $\mathscr{L}$ be a complete,* *orthocomplemented, weakly modular sublattice of* $\mathcal{P}(\mathcal{H})$ *with respect to the operations defined by* (2.1), (2.2) *and* (2.4). *Then there exists a subspace* $\mathcal{H}_1 = P_1\mathcal{H}$ *of* $\mathcal{H}$ *such that*

$$\mathcal{L}_1 = \{P_1 P P_1 \mid_{\mathcal{H}_1}; P \in \mathcal{L}\}$$

is a complete, orthocomplemented, weakly modular sublattice of $\mathcal{P}(\mathcal{H}_1)$, *isomorphic to* $\mathcal{L}$, *and which satisfies*

$$\mathbf{1}_{\mathcal{H}_1} \in \mathcal{L}_1 \subset \mathcal{P}(\mathcal{H}_1)$$

we have then $\mathcal{L}'' = \mathcal{L}'' \times \mathbf{C}\mathbf{1}_{\mathcal{H}\ominus\mathcal{H}_1}$, *hence* $P(\mathcal{L}'') = P(\mathcal{L}_1'') \times \{0, \mathbf{1}\}_{\mathcal{H}\ominus\mathcal{H}_1}$.

Proof. Define P_1 to be

$$P_1 = \bigvee_{P \in \mathcal{L}} P.$$

Since for any P in $\mathcal{L}: P < P_1$, we have

$$P_1 P P_1 \mid_{P_1\mathcal{H}} = P \mid_{P_1\mathcal{H}} \quad \text{and} \quad P \mid_{\mathcal{H}\ominus P_1\mathcal{H}} = 0$$

From this one sees immediately that $\mathcal{L}_1$ defined by

$$\mathcal{L}_1 = \{P_1 P P_1 \mid P_1\mathcal{H}; P \in \mathcal{L}\} \text{ is isomorphic to } \mathcal{L}.$$

Moreover

$$\mathbf{1}_{\mathcal{H}_1} = P_1 \mid_{P_1\mathcal{H}} \in \mathcal{L}_1$$

The last statement is a consequence of [8, §2.1].

In the following we usually will assume that $\mathbf{1}_{\mathcal{H}} \in \mathcal{L}$. If the lattice $\mathcal{L}$ is a distributive or Boolean one, the question formulated above is answered by a special case of Bade's theorem (see [5]; [10] p. 2214).

2.3 Theorem. *Let* $\mathcal{H}$ *be a complex Hilbertspace; let* $\mathcal{L}$ *be a Boolean sublattice of* $\mathcal{P}(\mathcal{H})$, *such that*

$$\mathbf{1}_{\mathcal{H}} \in \mathcal{L}$$

Then $P(\mathcal{L}'') = \mathcal{L}$.

Remark that no atomicity is required for $\mathcal{L}$ in Theorem 2.3. We will restrict ourselves to the case of propositional subsystems of $\mathcal{P}(\mathcal{H})$, which implies that we will only treat atomic lattices. There does exist however a class of propositional subsystems of $\mathcal{P}(\mathcal{H})$, which contain the identity but for which the conjecture (2.7) is false. Take for instance $\mathcal{H} = \mathbf{C}^4$, with canonical basis $\{e_1, e_2, e_3, e_4\}$. We define the lattice $\mathcal{L}$ by its atoms:

$$A(\mathcal{L}) = \left\{P_e; P_e = \text{Proj}_{\mathbf{C}e} \text{ with } e \text{ of the form } \sum_{i=1}^{4} \lambda_i e_i \quad \text{for some } \lambda_i \text{ in } \mathbf{R}\right\}$$

The lattice $\mathcal{L}$ generated by this set is an irreducible propositional system containing the identity. It is obvious that $\mathcal{L}'' = \mathcal{L}(\mathcal{H})$, hence $P(\mathcal{L}'') = \mathcal{P}(\mathcal{H})$ though $J \supsetneqq \mathcal{P}(\mathcal{H})$, which implies $\mathcal{L} \supsetneqq P(\mathcal{L}'')$.

We can however exclude these cases by a supplementary condition. We know, by Piron's representation theorem, that this lattice is isomorphic to the lattice of all closed subspaces of a V, $\mathbf{K}$, ϕ. In the present case it is clear that $\mathbf{K}=\mathbf{R}$. In the following we will restrict our attention to cases where $\mathbf{K}=\mathbf{C}$, thus excluding pathological cases like this one. Though this may seem a strong restriction, it is an absolutely necessary one, due to the fact that one usually defines von Neumann algebras as complex algebras of operators on a complex Hilbertspace.

To begin with we will restrict ourselves to irreducible propositional systems; the results obtained will be generalized subsequently.

3. A necessary and sufficient condition

We will now proceed to prove our main theorem. To do this we will make an intensive use of the material contained in [6]. For convenience we recall the different results which will be applied here.

First of all: a definition. A c-morphism f from $\mathcal{P}(\mathcal{H})$ to $\mathcal{P}(\mathcal{H}')$ is called an m-morphism if: $\forall \bar{x}, \bar{y}$ atoms in $\mathcal{P}(\mathcal{H})$: the subspace $f(\bar{x})+f(\bar{y})$ of $\mathcal{P}(\mathcal{H}')$ is closed. This is not the original definition given in [6], but according to Proposition 2.5 in [6], it is equivalent with it. We will use the definition given above instead of the original one because it turns out to be better adapted to the context in this paper. The following theorem was proven in [6]:

3.1 Theorem (see [6, Theorem 3.1]). *Let $\mathcal{H}$, $\mathcal{H}'$ be complex Hilbertspaces with dimension greater than two. Let f be a non-zero m-morphism mapping $\mathcal{P}(\mathcal{H})$ into $\mathcal{P}(\mathcal{H}')$. Then for every two non-zero vectors x, y in $\mathcal{H}$, there exists a bijective linear map F_{xy} from $f(\bar{y})$ to $f(\bar{x})$, such that the set $[F_{xy}; x, y\in\mathcal{H}, x\neq 0\neq y\}$ has the following properties:*

$$F_{xy}F_{yz}=F_{xz} \tag{3.1}$$

$$F_{xx}=\mathbf{1}_{f(\bar{x})} \tag{3.2}$$

$$F_{yx} \text{ is an isomorphism if } \|x\|=\|y\| \tag{3.3}$$

$$F_{\lambda x\,\lambda y}=F_{xy} \tag{3.4}$$

For every non-zero x in $\mathcal{H}$, there exist two orthogonal projections $P_1^{\bar{x}}$, $P_2^{\bar{x}}$ in $\mathcal{L}(f(\bar{x}))$ such that

$$P_1^{\bar{x}}+P_2^{\bar{x}}=f(\bar{x}) \tag{3.5}$$

$$F_{\lambda x\,x}=\lambda P_1^{\bar{x}}+\bar{\lambda}P_2^{\bar{x}} \tag{3.6}$$

Actually the theorem in [6] contains more than only this, but this is all we will need here.

If for one non-zero x in $\mathcal{H}$ the projection $P_2^{\bar{x}}(P_1^{\bar{x}})$ mentioned in (3.5) and (3.6) is zero, then all the $P_2^{\bar{y}}(P_1^{\bar{y}})$ are zero and f is called a linear (anti-linear) m-morphism. A general m-morphism is always the combination of a linear and an anti-linear m-morphism; if either of these two is non-trivial, the m-morphism is called mixed (see [6, Theorem 3.10 and Definition 3.9]).

We will also need the following rather weak consequence of Theorem 3.1 (see [6, Corollary 4.2]).

3.2 Theorem. *Let $\mathcal{H}$, $\mathcal{H}'$ be complex Hilbertspaces with dimension greater than two. Let f be a c-morphism mapping $\mathcal{P}(\mathcal{H})$ into $\mathcal{P}(\mathcal{H}')$ such that*

$f(\mathbf{1}_{\mathcal{H}})=\mathbf{1}_{\mathcal{H}'}$

$\exists x\in\mathcal{H}, x\neq 0$ *such that $f(\bar{x})$ is one-dimensional.*

Then $f(\mathcal{P}(\mathcal{H}))=\mathcal{P}(\mathcal{H}')$.

One more remark: from the construction of the F_{xy} (see the proof of Lemma 3.2 in [6]) one can infer the operator $F_{[(y,x)/\|y\|^2]yx}$ is in fact the restriction to $f(\bar{x})$ of the orthogonal projection on $f(\bar{y})$. Using all this, we can prove the following:

3.3 Theorem. *Let $\mathcal{H}$ be a complex Hilbertspace; let $\mathcal{L}$ be an irreducible propositional subsystem of $\mathcal{P}(\mathcal{H})$ such that*

$\mathbf{1}_{\mathcal{H}}\in\mathcal{L}$

$\exists\hat{\mathcal{H}}$ *complex Hilbertspace, $\dim\hat{\mathcal{H}}\geqslant 3$, and a c-isomorphism φ mapping $\mathcal{L}$ onto $\mathcal{P}(\hat{\mathcal{H}})$. Let i be the canonical injection mapping $\mathcal{L}$ into $\mathcal{P}(\mathcal{H})$. Then $\mathcal{L}=P(\mathcal{L}'')$ iff the c-morphism $f=i\circ\varphi$ is a non-mixed m-morphism.*

For the sake of comprehensibility, we have split the proof into different lemmas.

3.4 Lemma. *Let $\mathcal{H}$, $\hat{\mathcal{H}}$ be complex Hilbertspaces with $\dim\hat{\mathcal{H}}\geqslant 3$. Let f be a non-trivial m-morphism mapping $\mathcal{P}(\hat{\mathcal{H}})$ into $\mathcal{P}(\mathcal{H})$. we define*

$\mathcal{L}=f(\mathcal{P}(\hat{\mathcal{H}}))$

$A(\mathcal{L})=\{P\in\mathcal{L}; P$ *is an atom in* $\mathcal{L}\}$

If f is not mixed, then we can construct a set of partial isometries $\{U_{PQ}; P, Q\in A(\mathcal{L})\}\subset\mathcal{L}(\mathcal{H})$ such that

(1) $\forall P, Q\in A(\mathcal{L}): U_{PQ}$ *has initial subspace $Q\mathcal{H}$, and final subspace $P\mathcal{H}$.*
(2) $\forall P, Q\in A(\mathcal{L}): U_{PQ}\in\mathcal{L}''$
(3) $\forall P, Q, R\in A(\mathcal{L}): U_{PQ}U_{QR}=U_{PR}$.

Proof. For each atom P in $\mathcal{L}$ we choose a normalized vector x_P in $\mathcal{H}$ such that $P=f(\bar{x}_P)$. For any two P, Q in $A(\mathcal{L})$ we define the linear operator U_{PQ} by

$$\begin{aligned} &U_{PQ}\mid_{Q\mathcal{H}}=F_{x_P x_Q}\\ &U_{PQ}(\mathcal{H}\ominus Q\mathcal{H})=0 \end{aligned} \tag{3.7}$$

It follows from Theorem 3.1, in particular from (3.3) that this U_{PQ} is a partial isometry with initial subspace $Q\mathcal{H}$ and final subspace $P\mathcal{H}$ (see [7], p. 197).

From the remark preceding Theorem 3.3 we see that

$$PQ\mid_{Q\mathcal{H}}=F_{(x_P,x_Q)x_P x_Q} \tag{3.8}$$

If f is not mixed, we have either

$$F_{(x_P,x_Q)x_P x_Q}=(x_P,x_Q)F_{x_P x_Q}\quad\text{if } f \text{ is linear,}$$

or

$$F_{(x_P,x_Q)x_P x_Q}=\overline{(x_P,x_Q)}F_{x_P x_Q}\quad\text{if } f \text{ is anti-linear.}$$

In both cases there exists a complex number α such that

$$PQ\,|_{Q\mathcal{H}} = \alpha F_{x_P x_Q}, \tag{3.9}$$

hence, from (3.7), $PQ = \alpha U_{PQ}$

If P and Q are not orthogonal, we have $\alpha \neq 0$, hence

$$U_{PQ} = \alpha^{-1} PQ \in \mathcal{L}''$$

If P and Q are orthogonal, there exists a $R \in A(\mathcal{L})$ (take f.i. $R = f(\overline{x_P + x_Q})$) such that $R \not\perp P$ and $R \not\perp Q$.

From (3.1) we see that

$$\begin{aligned} U_{PQ}\,|_{Q\mathcal{H}} &= F_{x_P x_Q} = F_{x_P x_R} F_{x_R x_Q} = \alpha^{-1} PR\,|_{R\mathcal{H}} \cdot \beta^{-1} RQ\,|_{Q\mathcal{H}} \\ &= \alpha^{-1}\beta^{-1} PRQ\,|_{Q\mathcal{H}} \text{ for some non-zero } \alpha, \beta. \end{aligned}$$

hence

$$U_{PQ} = \alpha^{-1}\beta^{-1} PRQ \in \mathcal{L}''$$

The last statement is again a consequence of (3.7) and (3.1).

3.5 Lemma. *Let $\mathcal{H}$, $\hat{\mathcal{H}}$, f, $\mathcal{L}$ and $A(\mathcal{L})$ be as in Lemma 3.4. Suppose that f is not mixed, and that $f(\mathbf{1}_{\hat{\mathcal{H}}}) = \mathbf{1}_{\mathcal{H}}$. Let P be an element of $A(\mathcal{L})$. Then there exists a complex Hilbertspace $\tilde{\mathcal{H}}$ and an isomorphism φ mapping $\mathcal{H}$ onto $\tilde{\mathcal{H}} \otimes P\mathcal{H}$ such that the isomorphism ϕ defined by*

$$\begin{aligned} \phi : \mathcal{L}(\mathcal{H}) &\to \mathcal{L}(\tilde{\mathcal{H}} \otimes P\mathcal{H}) \\ A &\mapsto \varphi \circ A \circ \varphi^{-1} \end{aligned}$$

maps $\mathcal{L}''$ onto $\mathcal{L}(\tilde{\mathcal{H}}) \otimes \mathbf{C}_{P\mathcal{H}}$ and $\mathcal{L}$ onto $P(\tilde{\mathcal{H}}) \otimes \mathbf{1}_{P\mathcal{H}}$.

Proof. Since $\mathbf{1}_{\mathcal{H}}$ is an element of $\mathcal{L}$, there exists a set of orthogonal atoms $(P_i)_{i \in I}$ in $\mathcal{L}$, containing P, such that

$$\bigvee_{i \in I} P_i = \mathbf{1}_{\mathcal{H}}$$

(this can be obtained by a simple Zornication). Let i_0 be the element of I for which $P = P_{i_0}$. From Lemma 3.4 we know that for each i, j in I there exists a partial isometry $U_{ij} = U_{P_j P_i}$ in $\mathcal{L}''$ with initial subspace $P_i\mathcal{H}$ and final subspace $P_j\mathcal{H}$. This implies (see [8], p. 25) that there exists a Hilbertspace $\tilde{\mathcal{H}}$ and an isomorphism $\varphi : \mathcal{H} \to \tilde{\mathcal{H}} \otimes P\mathcal{H}$, such that the map

$$\begin{aligned} \phi : \mathcal{L}(\mathcal{H}) &\to \mathcal{L}(\tilde{\mathcal{H}} \otimes P\mathcal{H}) \\ A &\to \varphi \circ A \circ \varphi^{-1} \end{aligned}$$

maps $\mathcal{L}''$ bijectivily onto $\mathcal{L}(\tilde{\mathcal{H}}) \otimes \mathcal{L}''_P$ (we follow Dixmier's notations). On the other hand we have $\mathcal{L}'' = (A(\mathcal{L}))''$. Define the set $B(\mathcal{L})$ to be

$$B(\mathcal{L}) = \{Q_1 \cdots Q_n;\, n \in \mathbf{N},\, Q_1, \ldots, Q_n \in A(\mathcal{L})\} \tag{3.10}$$

This is a selfadjoint set, stable for the multiplication, which satisfies $(B(\mathcal{L}))'' = \mathcal{L}''$. It follows then from [8, §2.1, Proposition 1] that

$$\mathcal{L}''_P = (B(\mathcal{L})_P)'' \tag{3.11}$$

Let $Q_1, \ldots, Q_n$ be elements of $A(\mathscr{L})$. From (3.9) we see that either

$$PQ_1 \cdots Q_nP = 0$$

or

$$PQ_1 \cdots Q_nP = \alpha^{-1}U_{PQ_1}U_{Q_1Q_2} \cdots U_{Q_nP} \text{ for some } \alpha \in \mathbf{C}\backslash\{0\}.$$

In the latter case we have

$$PQ_1 \cdots Q_nP = \alpha^{-1}U_{PQ_2} \cdots U_{Q_nP} = \alpha^{-1}U_{PP} = \alpha^{-1}P$$

which implies that in both cases there exists a complex number β such that

$$PQ_1 \cdots Q_nP = \beta P \tag{3.12}$$

From (3.12) and the definition (3.10) of $B(\mathscr{L})$ we see that $B(\mathscr{L})_P \subset \mathbf{C}_{P\mathscr{H}}$, hence, because of (3.11),

$$\mathscr{L}''_P = \mathbf{C}_{P\mathscr{H}}$$

This implies that

$$\phi(\mathscr{L}'') = \mathscr{L}(\tilde{\mathscr{H}}) \otimes \mathbf{C}_{P\mathscr{H}}, \tag{3.13}$$

hence

$$\phi(\mathscr{L}) \subset P(\tilde{\mathscr{H}}) \otimes \mathbf{1}_{P\mathscr{H}}. \tag{3.13}$$

It is obvious from the construction of $\tilde{\mathscr{H}}$ and ϕ that there exists a one-dimensional projection Q in $\mathscr{L}(\tilde{\mathscr{H}})$ such that

$$\phi(P) = Q \otimes \mathbf{1}_{P\mathscr{H}}$$

Moreover we have $\phi(\mathbf{1}_{\mathscr{H}}) = \mathbf{1}_{\tilde{\mathscr{H}}} \otimes \mathbf{1}_{P\mathscr{H}}$. It follows from Theorem 3.2 that

$$\phi(\mathscr{L}) = \mathscr{P}(\tilde{\mathscr{H}}) \otimes \mathbf{1}_{P\mathscr{H}} \tag{3.14}$$

From (3.13) and (3.14) we see that

$$\begin{aligned}\phi(P(\mathscr{L}'')) = P(\phi(\mathscr{L}'')) &= \mathscr{P}(\tilde{\mathscr{H}}) \otimes \mathbf{1}_{P\mathscr{H}} \\ &= \phi(\mathscr{L})\end{aligned}$$

hence $P(\mathscr{L}'') = \mathscr{L}$ (ϕ is a bijective map).

This implies that the combination of Lemmas 3.4 and 3.5 proves that the condition mentioned in 3.3 is sufficient. The necessity of the condition will be proven in the lemma below.

3.6 Lemma. *Let $\mathscr{H}$, $\hat{\mathscr{H}}$ be complex Hilbertspaces with $\dim \hat{\mathscr{H}} \geq 3$. Let f be a unitary c-morphism mapping $\mathscr{P}(\hat{\mathscr{H}})$ into $\mathscr{P}(\mathscr{H})$. We define*

$$\mathscr{L} = f(\mathscr{P}(\hat{\mathscr{H}}))$$

$$A(\mathscr{L}) = \{P \in \mathscr{L}; P \text{ is an atom in } \mathscr{L}\}$$

If f is not an m-morphism, or if f is a mixed m-morphism, then

$$P(\mathscr{L}'') \supsetneqq \mathscr{L}$$

Proof. We suppose first that f is not an m-morphism. This implies that for some $P, Q \in A(\mathscr{L})$, $P\mathscr{H} + Q\mathscr{H}$ is not closed. Hence there exists no complex number α for which $PQP = \alpha P$. (The existence of such a complex number α would force the subspace $P\mathscr{H} + Q\mathscr{H}$ to be complete, hence closed.) But this implies $\mathscr{L}''_P \neq \mathbf{C}_{P\mathscr{H}}$, hence $\mathscr{L}''_P$ contains non-trivial projections of $\mathscr{L}(P\mathscr{H})$. Let S be such a projection. Define $S_1 \in \mathscr{L}(\mathscr{H})$ by

$$S_1 \mid_{P\mathscr{H}} = S$$
$$S_1(\mathscr{H} \ominus P\mathscr{H}) = 0$$

Because of the definition of $\mathscr{L}''_P$ (see [8], p. 16) this S_1 is a non-trivial projection of $\mathscr{L}''$, smaller then but different from P. Since P is an atom in $\mathscr{L}$, we see that S_1 is not contained in $\mathscr{L}$, although it is a projection of $\mathscr{L}''$. This implies

$$\mathscr{L} \subsetneqq P(\mathscr{L}'')$$

Suppose now that f is a mixed m-morphism. Take x, y in $\hat{\mathscr{H}}$ such that $\|x\| = \|y\| = 1$, and $(x, y) = \frac{1}{2}$. Define the atoms P, Q, R in $\mathscr{L}$ by

$$P = f(\bar{x}), \qquad Q = f(\bar{y}), \qquad R = f(\overline{x + iy})$$

Since (3.8) holds even when f is mixed, we have

$$PQRP \mid P\mathscr{H} = F_{(x,y)xy} F_{(y,(x+iy)/\sqrt{2})y[(x+iy)/\sqrt{2}]} F_{((x+iy)/\sqrt{2},x)[(x+iy)/\sqrt{2}]x}$$

Using (3.4) twice, we can reduce this to

$$\begin{aligned} PQRP \mid P\mathscr{H} &= F_{(x,y)(y,(x+iy)/\sqrt{2})((x+iy)/\sqrt{2},x)xx} \\ &= F_{(1/4)(i+(1/2))(1-(i/2))xx} \\ &= \frac{1}{4}\left(1 + \frac{3i}{4}\right)P_1^{\bar{x}} + \frac{1}{4}\left(1 - \frac{3i}{4}\right)P_2^{\bar{x}} \end{aligned}$$

Since both $P_1^{\bar{x}}$ and $P_2^{\bar{x}}$ are non-trivial, this implies that $PQRP\,|_{P\mathscr{H}}$ is not a multiple of $\mathbf{1}_{P\mathscr{H}}$, hence

$$\mathscr{L}''_P \neq \mathbf{C}_{P\mathscr{H}}$$

Using the same arguments as in the case where f is not an m-morphism, we see that this implies

$$P(\mathscr{L}'') \supsetneqq \mathscr{L}.$$

Remark. It is actually possible to prove a theorem equivalent to Theorem 3.3 without using explicitly the properties of the $\{F_{xy}\}$, except for the rather weak Theorem 3.2. This theorem has the advantage that it contains a necessary and sufficient condition which is easier to handle than the rather abstract one in Theorem 3.3. We will state this theorem without proving it here, but we indicate how the previous proofs should be modified to prove it. This reformulated theorem goes as follows:

3.7 Theorem. *Let $\mathscr{H}$ be a complex Hilbertspace; let $\mathscr{L}$ be an irreducible* *propositional subsystem of* $\mathcal{P}(\mathcal{H})$ *such that*

$$\mathbf{1}_{\mathcal{H}} \in \mathcal{L}$$

$\exists \hat{\mathcal{H}}$ *complex Hilbertspace,* $\dim \hat{\mathcal{H}} \geqslant 3$, *for which* $\mathcal{L}$ *is isomorphic to* $\mathcal{P}(\mathcal{H})$. *Then* $\mathcal{L} = P(\mathcal{L}'')$ *iff* $\forall P, Q, R$ *atoms in* $\mathcal{L}$, $\exists \alpha \in \mathbf{C}$ *such that*

$$PQRP = \alpha P \tag{3.15}$$

To prove this, one uses condition (3.15) to construct a set of $\tilde{U}_{PQ}$ which are in fact, up to some constant factor (depending on P and Q), the U_{PQ} defined by (3.7). These $\tilde{U}_{PQ}$ have again the properties stated in Lemma 3.4 except for the third one, which becomes:

$$\forall P, Q, R \in A(\mathcal{L}) : \exists \beta \in C, \qquad |\beta| = 1 \quad \text{such that} \quad \tilde{U}_{PQ}\tilde{U}_{QR} = \beta \tilde{U}_{PR}$$

Once these are constructed, the reasoning in the proof of Lemma 3.5 can be repeated to prove the sufficiency of condition (3.15). The necessity of condition (3.15) is proven by the same reasoning as the one used in the second half of the proof of Lemma 3.6. The equivalence of Theorem 3.3 and Theorem 3.7 is proven by the following:

3.8 Theorem. *Let* $\mathcal{H}$, $\hat{\mathcal{H}}$ *be complex Hilbertspaces, with* $\dim \mathcal{H} \geqslant 3$; *let f be a unitary c-morphism mapping* $\mathcal{P}(\hat{\mathcal{H}})$ *into* $\mathcal{P}(\mathcal{H})$. *Then:*

– *f is an m-morphism iff* $\forall P, Q$ *atoms in* $f(\mathcal{P}(\hat{\mathcal{H}})) : \exists \alpha \in \mathbf{C}$ *such that*

$$PQP = \alpha P \tag{3.16}$$

– *f is a non-mixed m-morphism iff the atoms in* $f(\mathcal{P}(\hat{\mathcal{H}}))$ *satisfy condition* (3.17).

Proof. The fact that (3.16) compels f to be an m-morphism was proven in the first part of the proof of Lemma 3.6. The other implication can be proven as follows: Let P, Q be two atoms in $f(\mathcal{P}(\hat{\mathcal{H}}))$, and let x, y be normalized vectors in $\hat{\mathcal{H}}$ such that

$$f(\bar{x}) = P \qquad f(\bar{y}) = Q$$

Because of (3.8) we have

$$\begin{aligned} PQP \mid_{P\mathcal{H}} &= F_{(x,y)xy}F_{(y,x)yx} = F_{(x,y)(y,x)xx} \\ &= |(x, y)|^2 \, \mathbf{1}_{f(\bar{x})} \end{aligned}$$

Hence

$$PQP = |(x, y)|^2 \, P.$$

We prove now the second statement in two steps. Suppose first that f is a non-mixed m-morphism. Let P, Q, R be three atoms in $f(\mathcal{P}(\hat{\mathcal{H}}))$, and let x, y, z be normalized vectors in $\hat{\mathcal{H}}$ satisfying $f(\bar{x}) = P$, $f(\bar{y}) = Q$, $f(\bar{z}) = R$. Applying (3.8), we obtain

$$\begin{aligned} PQRP \mid_{P\mathcal{H}} &= F_{(x,y)xy}F_{(y,z)yz}F_{(z,x)zx} \\ &= F_{(x,y)(y,z)(z,x)xx} \end{aligned} \tag{3.17}$$

Since f is not mixed we have either

$$PQRP\,|_{P\hat{\mathcal{H}}}=(x,y)(y,z)(z,x)\mathbf{1}_{f(\bar{x})}$$

or

$$PQRP\,|_{P\hat{\mathcal{H}}}=(x,z)(z,y)(y,x)\mathbf{1}_{f(\bar{x})}$$

In both cases there exists an α such that $PQRP=\alpha P$, which implies that (3.15) holds. Suppose now that condition (3.15) is satisfied. (3.16) being a special case of (3.15), this implies that f is an m-morphism. We can then again use the F_{xy} and their properties to obtain (3.17). If f were mixed, we could choose x, y, z as in the proof of Lemma 3.6, and we should obtain:

$$PQRP\,|_{P\hat{\mathcal{H}}}=\frac{1}{4}\left(1+\frac{3i}{4}\right)P_1^{\bar{x}}+\frac{1}{4}\left(1-\frac{3i}{4}\right)P_2^{\bar{x}},$$

where both $P_1^{\bar{x}}$ and $P_2^{\bar{x}}$ would be non-trivial. But this would imply that condition (3.15) is not satisfied, which is false. Hence f is not mixed.

Condition (3.15) gives us an easier criterion to decide whether the main theorem is applicable or not. It is interesting to note that condition (3.15) contains both conditions that f should be an m-morphism and that f should not be mixed. On the other hand, the proof that (3.15) is a necessary condition holds independently of the dimension of $\hat{\mathcal{H}}$, while the properties of the F_{xy} can only be used if $\dim\hat{\mathcal{H}}\geq 3$. These properties are used several times in the preceding proofs, which implies that the restriction $\dim\hat{\mathcal{H}}\geq 3$ plays a vital role in Theorem 3.8 as well as in Theorem 3.3. Since we have to use Theorem 3.2 to prove Theorem 3.7, the same is true for Theorem 3.7. We will give here two counterexamples with $\dim\hat{\mathcal{H}}=2$. In the first one we have $P(\mathcal{L}'')\supsetneqq\mathcal{L}$ although condition (3.15) is satisfied; in the second one we show that the first statement in Theorem 3.8 does not hold any more if $\dim\hat{\mathcal{H}}=2$.

3.9 Counterexample. Take $\mathcal{H}=\mathbf{C}^4$, $\hat{\mathcal{H}}=\mathbf{C}^2$. The one-dimensional projection operators in $\mathcal{L}(\mathbf{C}^2)$ are given by

$$P_{\theta\varphi}=\overline{e_{\theta\varphi}}\quad\text{with}\quad e_{\theta\varphi}=\cos\theta e_1+e^{i\varphi}\sin\theta e_2$$

$$\theta\in\left[0,\frac{\pi}{2}\right],\qquad \varphi\in[0,2\pi]$$

where e_1, e_2 is the standard basis in $\mathbf{C}^2$. Let f_1, f_2, f_3, f_4 be the standard basis in $\mathbf{C}^4$. We define

$$\begin{aligned}f_{\theta\varphi}&=\cos\theta f_1+e^{i\varphi}\sin\theta f_3\\ g_{\theta\varphi}&=\cos\theta f_2+e^{i\varphi}\sin\theta f_4\end{aligned}\qquad \theta\in\left[0,\frac{\pi}{2}\right],\qquad \varphi\in[0,2\pi]$$

We define now a map f from $\mathcal{P}(\mathbf{C}^2)$ to $\mathcal{P}(\mathbf{C}^4)$ by

$$f(0)=0\qquad f(\mathbf{1}_{\mathbf{C}^2})=\mathbf{1}_{\mathbf{C}^4}$$

$$f(P_{\theta\varphi})=\begin{cases}\mathrm{Proj}_{\mathrm{Span}(f_{\theta(\varphi/4)},\,g_{\theta(\varphi/4)})} & \text{if}\quad \varphi\in[0,\pi[\\ \mathrm{Proj}_{\mathrm{Span}(f_{\theta[(\varphi+3\pi)/4]},\,g_{\theta[(\varphi+3\pi)/4]})} & \text{if}\quad \varphi\in[\pi,2\pi[\end{cases}$$

It is almost trivial to check that this f is a unitary c-morphism, which implies that $\mathscr{L}=f(\mathscr{P}(\mathbf{C}^2))$ is a sublattice of $\mathscr{P}(\mathbf{C}^4)$, isomorphic with $\mathscr{P}(\mathbf{C}^2)$, which contains $\mathbf{1}_{\mathbf{C}^4}$. We will denote the atoms $f(P_{\theta\varphi})$ of $\mathscr{L}$ by $Q_{\theta\varphi}$. We have now

$$Q_{\theta\varphi}x = f_{\theta\tilde{\varphi}}(f_{\theta\tilde{\varphi}}, x) + g_{\theta\tilde{\varphi}}(g_{\theta\varphi}, x) \quad \text{where} \quad \tilde{\varphi} = \frac{\varphi}{4} + \frac{3\pi}{4}\chi_{[\pi, 2\pi[}(\varphi)$$

using this and the fact that $(f_{\theta\varphi}, f_{\theta'\varphi'}) = (g_{\theta\varphi}, g_{\theta'\varphi'})$ for any θ, θ', φ, φ', we see that

$$Q_{\theta\varphi}Q_{\theta'\varphi'}Q_{\theta''\varphi''}Q_{\theta\varphi} = (f_{\theta\tilde{\varphi}'}, f_{\theta'\tilde{\varphi}'})(f_{\theta'\tilde{\varphi}'}, f_{\theta''\tilde{\varphi}''})(f_{\theta''\tilde{\varphi}''}, f_{\theta\tilde{\varphi}})Q_{\theta\varphi},$$

which implies that condition (3.15) is satisfied. On the other hand, we know that $\mathscr{L}$ contains the projections

$$Q_{00} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Q_{(\pi/4)0} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

which implies

$$\mathscr{L}' \subset \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}; \qquad A \in \mathscr{L}(\mathbf{C}^2) \right\}.$$

But this implies that the projection operator

$$\frac{1}{2}\begin{pmatrix} 1 & 0 & i & 0 \\ 0 & 1 & 0 & -i \\ -i & 0 & 1 & 0 \\ 0 & i & 0 & 1 \end{pmatrix}$$

is contained in $\mathscr{L}''$, although one can check that it is not an element of $\mathscr{L}$. We have thus $\mathscr{L} \subsetneqq P(\mathscr{L}'')$, although condition (3.15) is satisfied. This is due to the fact that $\mathscr{L}$ is isomorphic to a $\mathscr{P}(\hat{\mathscr{H}})$ where $\hat{\mathscr{H}}$ has dimension smaller than three.

3.10 Counterexample. Take again $\mathscr{H} = \mathbf{C}^4$, $\hat{\mathscr{H}} = \mathbf{C}^2$. For any θ in $[0, \pi/2]$, define $\hat{\theta}$ by $\hat{\theta} = (2 - \cos 2\theta)\pi/4$. With the same notations as in the previous counterexample, we define a map f from $\mathscr{P}(\mathbf{C}^2)$ to $\mathscr{P}(\mathbf{C}^4)$ by

$$f(0) = 0 \qquad f(\mathbf{1}_{\mathbf{C}^2}) = \mathbf{1}_{\mathbf{C}^4}$$

$$f(P_{\theta\varphi}) = R_{\theta\varphi} = \mathrm{Proj}_{\mathrm{Span}(f_{\theta\varphi}, g_{\hat{\theta}\varphi})}$$

One can check (see also [6]) that this map is a unitary c-morphism, i.e. that $\mathscr{L} = f(\mathscr{P}(\mathbf{C}^2))$ is a sublattice of $\mathscr{P}(\mathbf{C}^4)$, which contains $\mathbf{1}_{\mathbf{C}^4}$ and which is isomorphic to $\mathscr{P}(\mathbf{C}^2)$; A simple calculation yields

$$R_{0\varphi}R_{\theta\varphi}R_{0\varphi} = \begin{pmatrix} \cos^2\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \cos^2\hat{\theta} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

for $\theta \notin \{0, \pi/4, \pi/2\}$, this is clearly not a multiple of $R_{0\varphi}$, which implies that not only (3.15), but also (3.16) is not satisfied. One can immediately check that

$$\bar{f}_3 = \frac{1}{\cos^2 \hat{\theta} - \cos^2 \theta} (Q_{0\varphi} Q_{\theta\varphi} Q_{0\varphi} - \cos^2 \theta Q_{0\varphi})$$

is an element of $\mathscr{L}'' \backslash \mathscr{L}$, which implies $P(\mathscr{L}'') \supsetneqq \mathscr{L}$. This is a natural consequence of the fact that the necessary condition (3.15) is not satisfied. On the other hand, we see that, though f is an m-morphism, (3.16) is not satisfied: this is due to the fact that $\mathscr{L}$ is isomorphic to a $\mathscr{P}(\hat{\mathscr{H}})$ with $\dim \hat{\mathscr{H}} < 3$.

4. Generalization to reducible propositional systems

Up till now, we have restricted ourselves to the case where the propositional system is irreducible. This is however not a crucial point, and we will now consider the general case. We first remind the reader of some results obtained by C. Piron (see f.i. [1], Theorem 2.37).

4.1 Theorem. *Let $\mathscr{L}$ be a propositional system, Z its center. Then Z is an atomic Boolean sublattice of $\mathscr{L}$. If $(P_j)_{j \in J}$ is a maximal set of stoms in Z, we can write $\mathscr{L}$ as the direct union of the segments $[0, P_j]$*

$$\mathscr{L} = \bigvee_{j \in J} [0, P_j]$$

Remarks

1. If P, Q are two different atoms in the lattice Z, then P and Q are orthogonal (i.e. $P < Q'$), for they commute with each other:
2. For each atom P in Z, the segment $[0, P]$ is an irreducible sublattice of $\mathscr{L}$. Using Theorem 4.1 we can prove the following:

4.2 Theorem. *Let $\mathscr{H}$ be a complex Hilbertspace; let $\mathscr{L}$ be a propositional subsystem of $\mathscr{H}$ such that $\mathscr{H} \in \mathscr{L}$. Let $(P_j)_{j \in J}$ be the set of atoms in Z, the central sublattice of $\mathscr{L}$. Suppose that for each $j \in J$, there exists a complex Hilbertspace $\hat{\mathscr{H}}_j$, $\dim \hat{\mathscr{H}}_j \geqslant 3$, and an isomorphism φ_j mapping $[0, P_j]$ onto $P(\hat{\mathscr{H}}_j)$. Let $i_j : [0, P_j] \to \mathscr{P}(\mathscr{H})$ be the canonical injection of $[0, P_j]$ into $\mathscr{P}(\mathscr{H})$. Then $P(\mathscr{L}'') = \mathscr{L}$ iff $\forall j \in J : i_j \circ \varphi_j^{-1}$ is a non-mixed m-morphism.*

Proof. Since the P_j are orthogonal, and

$$\bigvee_{j \in J} P_j = \mathbf{1}_{\mathscr{H}},$$

we have

$$\mathscr{H} = \bigoplus_{j \in J} P_j \mathscr{H}$$

Since all the P_j are elements of $\mathscr{L}' \cap \mathscr{L}$ (they commute with $\mathscr{L}$), we have (see [8], p. 20)

$$\mathscr{L}'' = \prod_{j \in J} \mathscr{L}''_{P_j}$$

So if Q is a projection in $\mathscr{L}''$, there exist a family of projections $(Q_j)_{j\in J}$ with $Q_j \in \mathscr{L}''_{P_j}$ such that

$$Q = \sum_{j\in J} Q_j = \bigvee_{j\in J} Q_j.$$

Applying Theorem 3.3 to the propositional subsystems $[0, P_j]$ of $\mathscr{P}(P_j\mathscr{H})$ yields

$$[0, P_j] = P(\mathscr{L}''_{P_j})$$

hence $Q_j \in [0, P_j] \subset \mathscr{L}$ for each j in J. This implies

$$Q = \bigvee_{j\in J} Q_j \in \mathscr{L},$$

hence $P(\mathscr{L}'') \subset \mathscr{L}$, which implies $P(\mathscr{L}'') = \mathscr{L}$. The necessity of the condition can be proven as in Lemma 3.6. Indeed, suppose that

$i_k \circ \varphi_k^{-1} : \mathscr{P}(\hat{\mathscr{H}}_k) \to \mathscr{P}(\mathscr{H})$ is not a linear or anti-linear f-morphism.

Then (see the proof of Lemma 3.6) it is possible to find atoms Q_1, Q_2, Q_3 in $\mathscr{L}$ which are smaller than P_k and for which $Q_1Q_2Q_3Q_1$ is not a multiple of Q_1. (If f is not an f-morphism, one can even choose Q_2 and Q_3 to be equal). From Theorem 3.7 one infers that

$$P(\mathscr{L}''_{P_k}) \supsetneqq [0, P_k].$$

This implies that a projection exists for which

$$R \in \mathscr{L}''_{P_k} \qquad R \notin [0, P_k]$$

The projection $\hat{R}$ defined by $\hat{R} \mid P_k\mathscr{H} = R$, $\hat{R} \mid \mathscr{H} \ominus P_k\mathscr{H} = 0$ is then clearly an element of $\mathscr{L}''$ and not of $\mathscr{L}$:

$$\hat{R} \in P(\mathscr{L}'') \backslash \mathscr{L}$$

Remark. One can again replace the necessary and sufficient condition in Theorem 4.2 by the condition

$$\forall P, Q, R \text{ atoms in } \mathscr{L} : \alpha \in \mathbf{C} \quad \text{such that } PQRP = \alpha P \tag{4.1}$$

Indeed, it is obvious that any atom is contained in a $[0, P_j]$. If the P, Q, R belong to different $[0, P_j]$, condition (4.1) is automatically satisfied with $\alpha = 0$. If we write (4.1) for all the P, Q, R smaller than the same P_j, we see from Theorem 3.8 that we get a condition equivalent to the one in Theorem 4.2. On the other hand Theorem 3.7 can be proven using only Theorem 3.2, and not Theorem 3.1 (see the remark made previously). Since the dimension condition $\dim \hat{\mathscr{H}} \geq 3$ in Theorem 3.2 can be replaced by $\dim \hat{\mathscr{H}} \neq 2$ (for $\dim \hat{\mathscr{H}} = 1$ Theorem 3.2 is trivial), we can reformulate Theorem 4.2 as follows:

4.3 Theorem. *Let $\mathscr{H}$ be a complex Hilbertspace, $\mathscr{L}$ a propositional subsystem of $\mathscr{P}(\mathscr{H})$; let Z be its central lattice, and $A(\mathscr{L})$ the set of its atoms. Let $(P_j)_{j\in J}$ be the set of atoms of Z, and suppose that for any $j \in J$ there is a complex Hilbertspace $\hat{\mathscr{H}}_j$ with $\dim \hat{\mathscr{H}}_j \neq 2$ such that $[0, P_j]$ is isomorphic with $\mathscr{P}(\hat{\mathscr{H}}_j)$. Then $\mathscr{L} = P(\mathscr{L}'')$ iff $\forall P, Q, R \in A(\mathscr{L}) : \exists\, \alpha \in \mathbf{C}$ such that $PQRP = \alpha P$.*

One sees immediately that in this general form, the theorem can be applied to Boolean atomic lattices, which yields a trivial special case of Bade's theorem.

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