Using Fredholm determinants to estimate the smoothness of refinable functions

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Abstract. The regularity of refinable functions is linked to the spectral properties of special operators associated to the refinement equation; we then use the Fredholm determinant theory to derive numerical estimates for the spectral radius of these operators in certain spaces. This new technique is particularly useful for estimating the regularity in the cases where the refinement equation has an infinite number of nonzero coefficients and in the multidimensional cases.

§1 Introduction

This paper reports on joint work with Albert Cohen; a longer version will appear elsewhere (see [5]). Refinable functions are functions that satisfy a refinement equation, *i.e.*,

$$\varphi(x) = 2\sum_n h_n \varphi(2x-n) \tag{1.1}$$

The coefficients h_n are often, but not always, chosen finite in number. Such functions appear in different settings, most notably in subdivision schemes for computer aided design, where they are tools for the fast generation of smooth curves and surfaces (see [1] and [13] for reviews), and in the construction of wavelet bases and multiresolution analysis (see [24], [7], and [25]).

The regularity of a function φ can be measured in different ways; we shall concentrate here on Sobolev exponents. If φ is in C^n but not in C^{n+1}, then its Hölder exponent is given by $\mu = n+\nu$ with

$$\nu = \inf_x \left(\liminf_{|t|\to 0} \frac{\log|\varphi^{(n)}(x+t)-\varphi^{(n)}(x)|}{\log|t|} \right) . \tag{1.2}$$

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The Sobolev exponent s of φ is defined by

$$s = \sup\left\{\gamma\,;\,\int |\hat{\varphi}(\omega)|^2(1+|\omega|^2)^\gamma d\omega \,<\, +\infty\right\} \tag{1.3}$$

where $\hat{\varphi}(\omega) = \int \varphi(x)e^{-i\omega x}dx$ is the Fourier transform of φ. One can generalize this to L^p-Sobolev exponents s_p which, following [20], we define by

$$s_p = \sup\left\{\gamma\,;\,\int |\hat{\varphi}(\omega)|^p(1+|\omega|^p)^\gamma d\omega \,<\, +\infty\right\} \tag{1.4}$$

These different regularity indices are related to each other by $s = s_2$, $\mu \geq s_1$ and, by Hölder's inequality, $s_r - r^{-1} \leq s_p - p^{-1}$ for $0 \leq p \leq r \leq 2p$. We shall restrict ourselves here mostly to $p = 2$, as in (1.3); for more general p, see [20] or [5].

Most of the techniques developed to estimate the regularity of a refinable function concentrate on the case where only finitely many h_n are non-zero, which was, until recently, the only case of interest for applications: in subdivision schemes it corresponds to finite masks; in wavelet constructions, to compactly supported scaling functions and wavelets. If there are only $N+1$ nonzero h_n, then Micchelli and Prautzsch [26], and Daubechies and Lagarias [9,10] showed how to find, at least in principle, the Hölder exponent of φ by computing bounds on the norms of $N \times N$ matrices; in practice, this method becomes quickly impractical if N is not small. Still for the case of finitely many nonzero h_n, a technique that can handle larger N was proposed by Rioul [28], and Dyn and Levin [14]; it is still the best available technique for finding the Hölder exponent when N is finite. In general it is easier to compute the Sobolev exponents; these can moreover be used to find a bound on the Hölder exponent, since $\mu \geq s_1 \geq s_2 + 1/2$. In the case where $m(\omega) = \sum_n h_n e^{-in\omega}$ is a nonnegative trigonometric polynomial, one even has $\mu = s_1$, which was exploited in one of the first computations of the regularity of a refinable function in [11] for the special case $h_0 = 1/2$, $h_{\pm 1} = 9/32$, $h_{\pm 3} = -1/32$, all other $h_n = 0$, related to Lagrangian interpolation and later generalized in [12]. When $m(\omega)$ is not restricted to be nonnegative, most of the first developments were concentrated on the computation of $s_2 = s$; see [6], [15], and the appendix in [7]. Extensions to the computation of s_p (including $p = 1, 2$) have appeared in [18], [34], and [20]. With the exception of [20], all the papers above apply only to N finite. Most of them are also hard to generalize to the multidimensional case where (1.1) is replaced by

$$\varphi(x) = |D| \sum_n h_n \varphi(Dx - n) \;, \tag{1.5}$$

where $n \in \mathbf{Z}^d$, $x \in \mathbb{R}^d$ and D is a $d \times d$ matrix with integer entries and all eigenvalues strictly superior to 1 in absolute value; $|D|$ is the determinant of D. Examples of the type (1.5) occur in *e.g.*, wavelet bases corresponding to quincunx subsampling in two dimensions proposed for image processing in [33], [16], and [21], with explicit orthonormal wavelet bases in [17], [22], and [4]. As illustrated by the trickiness of the estimates in [4] and especially in [35], it is not easy to find the regularity in the multidimensional case by generalizing the approach referred to above, even when only finitely many h_n are nonzero.

We present here a different technique for computing s_p for a refinable function. This technique is independent of whether the h_n are finite in number or not (like in [20]) and it generalizes easily to the multidimensional case. Like most of the other approaches, our results hinge on the computation of the spectral radius of a particular operator (see §3 and §4 below). We introduce a different space on which this operator acts however, and we use a computation of the Fredholm determinant borrowed from [29] to compute its spectral radius.

§2 A primer on Fredholm determinants

This section presents the results on Fredholm determinants of trace class operators on Hilbert spaces that will be needed in the sequel. We shall work in a (generic) separable Hilbert space denoted by $\mathcal{H}$. Recall that any bounded operator A on $\mathcal{H}$ can be written as $A = U(A^*A)^{1/2} = U|A|$, where U is a partial isometry with $\|Ux\| = \|x\|$ if $x \in \overline{\text{Ran } |A|}$, $Ux = 0$ if $x \perp \text{Ran } |A|$. If A is a compact operator, then so is $|A|$; the spectrum of $|A|$ then consists of a decreasing sequence of nonnegative eigenvalues. The strictly positive eigenvalues λ_m of $|A|$ are the singular values of A. If φ_m is a corresponding orthonormal system of eigenvectors of $|A|$, and we define another orthonormal system ψ_m by $\psi_m = U\varphi_m$, then we have the standard singular value decomposition for A:

$$Ax = \sum_m \lambda_m \langle x, \varphi_m \rangle \psi_m \ . \tag{2.1}$$

A compact operator A is called *trace class* if $\sum_m \lambda_m < \infty$. This is equivalent to the requirement that

$$\sum_n |\langle Ae_n, e_n \rangle| < \infty \text{ for all orthonormal systems } \{e_n\} \ .$$

If A is trace-class, then the sum of the series $\sum_n \langle Ae_n, e_n \rangle$ is independent of the choice of the $\{e_n\}$, and is called the trace of A:

$$\text{Tr } A = \sum_n \langle Ae_n, e_n \rangle \ .$$

In particular, using the representation (2.1) above, one has

$$\text{Tr } |A| = \sum_m \langle |A|\varphi_m, \varphi_m \rangle = \sum_m \lambda_m \ . \tag{2.2}$$

In [23] it is proved that for any trace-class operator A in a Hilbert space $\mathcal{H}$ one has

$$\text{Tr } A = \sum_n \alpha_n \tag{2.3}$$

where the α_n are the non-zero eigenvalues of A, taken with their algebraic multiplicity.

The trace class operators form an ideal in the algebra of bounded operators: the product of a trace class operator and a bounded operator is again trace class. Consequently all the powers $A^n, n \geq 1$, of a trace class operator are trace class as well. One has

$$\text{Tr } A^n = \sum_k \alpha_k^n \ . \tag{2.4}$$

Finally, note that

$$\begin{aligned} \sum_k |\alpha_k| &= \sum_k |\langle A u_k, u_k \rangle| \leq \sum_{k,m} \lambda_m |\langle u_k, \varphi_m \rangle| \, |\langle \psi_m, u_k \rangle| \\ &\leq \sum_m \lambda_m \left(\sum_k |\langle u_k, \varphi_m \rangle|^2 \right)^{1/2} \left(\sum_k |\langle \psi_m, u_k \rangle|^2 \right)^{1/2} \\ &\leq \sum_m \lambda_m \|\varphi_m\| \, \|\psi_m\| = \sum_m \lambda_m = \text{Tr } |A| \ . \end{aligned} \tag{2.5}$$

The Fredholm determinant of A is defined as

$$D_A(z) = \det \ (I - zA) = \prod_{n=1}^{+\infty} (1 - z\alpha_n) \ , \tag{2.6}$$

where the α_n occur with their multiplicity. Because $\sum_n |\alpha_n| < +\infty$, $D_A(z)$ is an entire function with its zeros exactly at the α_n^{-1}, with the same multiplicity. In particular, the spectral radius ρ_A of A is given by

$$\rho_A = \left(\min\{|z_0| \, ; \, D_A(z_0) = 0\}\right)^{-1} \ . \tag{2.7}$$

For sufficiently small values of z (*e.g.*, for $|z| < \rho_A^{-1}$), one can rewrite $D_A(z)$ as follows:

$$\begin{aligned} D_A(z) &= \exp \left[\sum_{n=1}^{\infty} \log(1 - z\alpha_n) \right] = \exp \left[\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} -\frac{1}{m} (z\alpha_n)^m \right] \\ &= \exp \left[- \sum_{m=1}^{\infty} \frac{1}{m} z^m \text{Tr } A^m \right] \ , \end{aligned} \tag{2.8}$$

a formula which already shows that D_A is completely determined by the traces Tr A^m. Expanding the infinite product in (2.6) leads to a different formula for $D_A(z)$,

$$D_A(z) = 1 + \sum_{k=1}^{+\infty} z^k \gamma_k, \tag{2.9}$$

with

$$\gamma_k = \sum_{l_1<l_2<\ldots<l_k} \alpha_{l_1} \ldots \alpha_{l_k}.$$

We then have

$$|\gamma_k| \leq \frac{1}{k!} \left(\sum_l |\alpha_l| \right)^k \leq \frac{1}{k!} (\mathrm{Tr}|A|)^k. \tag{2.10}$$

It follows that we can therefore always write

$$D_A(z) = D_A^N(z) + R_A^N(z), \tag{2.11}$$

where $D_A^N(z)$ is the Taylor series for D_A truncated after the term in z^N, and

$$|R_A^N(z)| \leq \sum_{k=N+1}^{+\infty} \frac{1}{k!} (\mathrm{Tr}|A|)^k \; |z|^k \;. \tag{2.12}$$

We shall use this estimate to find the smallest zero of D_A: since D_A^N is a polynomial, its smallest zero can be found by a host of different numerical methods, and the control we have via (2.12) on the rest term R_A^N will tell us that the smallest zero of D_A itself cannot be far from that of D_A^N if N is sufficiently large (see §5).

In order to identify the zeros of D_A^N, we need again a different representation; in particular, we are interested in a way of computing the Taylor coefficients of D_A which does not require knowledge of the eigenvalues α_n. To do this, let us start by restricting ourselves to the disk $B\left(0, \rho_A^{-1}\right)$. On this disk, we can write (using a trick going back to Newton)

$$\begin{aligned}
\sum_{k=0}^{\infty} (k+1)\gamma_{k+1} z^k &= D'_A(z) = \sum_{n=1}^{\infty} (-\alpha_n) \prod_{\substack{m=1 \\ m\neq n}}^{\infty} (1 - \alpha_m z) \\
&= -D_A(z) \sum_{n=1}^{\infty} \frac{\alpha_n}{1-\alpha_n z} = -D_A(z) \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \alpha_n^{k+1} z^k \\
&= -D_A(z) \sum_{k=0}^{\infty} \mathrm{Tr}\; A^{k+1} z^k = -\sum_{r=0}^{\infty} z^r \sum_{m=0}^{r} \gamma_m \mathrm{Tr}\; A^{r-m+1} \;,
\end{aligned}$$

where the reordering of the sums is allowed because the series converges absolutely for $|z| < (\rho_A)^{-1}$, and where we have introduced $\gamma_0 = 1$. It follows that

$$\gamma_{k+1} = \frac{-1}{k+1} \sum_{m=0}^{k} \gamma_m \mathrm{Tr}\, A^{k+1-m} \ . \tag{2.13}$$

§3 Transfer operators as trace-class operators

The operators to which we shall apply the results in the previous section act on 2π-periodic functions $f(\omega)$ and are defined by

$$(\mathcal{L}_w f)(\omega) = w\left(\frac{\omega}{2}\right) f\left(\frac{\omega}{2}\right) + w\left(\frac{\omega}{2}+\pi\right) f\left(\frac{\omega}{2}+\pi\right) \ , \tag{3.1}$$

where w is a 2π-periodic weight function, for which several concrete choices will be proposed in §4. We shall say that $\mathcal{L}_w$ is the *transfer operator* associated with the function w; these operators are also called Perron-Frobenius operators or transition operators in the literature. We shall always assume that the Fourier coefficients of w decay exponentially; that is,

$$w(\omega) = \sum_n w_n \ e^{-in\omega} \tag{3.2}$$

and

$$|w_n| \le C \ e^{-\gamma|n|} \tag{3.3}$$

for some $C, \gamma > 0$. In terms of the Fourier coefficients f_n of $f(\omega) = \sum_n f_n \ e^{-in\omega}$, (3.1) can also be rewritten as

$$(\mathcal{L}_w f)_n = \frac{1}{2\pi} \int_0^{2\pi} (\mathcal{L}_w f)(\omega) e^{in\omega} d\omega = 2 \sum_k w_{2n-k} f_k \ . \tag{3.4}$$

When no confusion is possible, we shall often drop the subscript w on $\mathcal{L}_w$.

Operators of the type (3.1) can be studied in many different function spaces. They have been linked with the study of refinable functions before; see *e.g.*, [6], [15], [34], [20]. They are special cases of the operators in [29,30]. In this section we discuss their properties on some Hilbert spaces of analytic functions.

As candidates for the space $\mathcal{H}$ we define

$$E_\alpha = \left\{ f \ 2\pi\text{−periodic; } f(\omega) = \sum_n f_n \ e^{-in\xi} \right. \\ \left. \text{and } \|f\|_\alpha^2 = \sum_n |f_n|^2 \ e^{2|n|\alpha} < \infty \right\} . \tag{3.5}$$

(Note that these are different from the spaces E^α in [20].) The E_α are Hilbert spaces of analytic functions ($f \in E_\alpha$ can be extended to complex $\omega = \omega_1 + i\omega_2$ and is then analytic for $|\omega_2| = |\text{Im}\ \omega| < \alpha$); their inner product is given by

$$\langle f, g\rangle_\alpha = \sum_n f_n \overline{g}_n\ e^{2|n|\alpha}\ . \tag{3.6}$$

Note that for each α, the constant function 1 is in E_α; moreover, for $f \in E_\alpha$,

$$\langle f, 1\rangle_\alpha = f_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\omega) d\omega\ . \tag{3.7}$$

The functions

$$e_{n,\alpha}(\omega) = e^{-|n|\alpha}\ e^{-in\omega} \tag{3.8}$$

constitute an orthonormal basis in E_α.

In order to be able to apply §2 to $\mathcal{L}$ and E_α, we need to verify that $\mathcal{L}$ is a bounded operator on E_α, and that $\mathcal{L}$ is trace class on E_α. We start by computing $\|\mathcal{L}f\|_\alpha$, using (3.4):

$$\begin{aligned}
\|\mathcal{L}f\|_\alpha^2 &= \sum_n |(\mathcal{L}f)_n|^2\ e^{2|n|\alpha} = 4\sum_n \left|\sum_k w_{2n-k} f_k\right|^2\ e^{2|n|\alpha} \\
&\leq 4\|f\|_\alpha^2 \sum_{n,k}\ |w_{2n-k}|^2\ e^{-2|k|\alpha}\ e^{2|n|\alpha} \\
&\leq 4C^2\|f\|_\alpha^2 \sum_{n,k}\ e^{-2(\alpha-\gamma)|k|}\ e^{-2(2\gamma-\alpha)|n|}\ ,
\end{aligned}$$

where the last inequality used (3.3) and $|2n-k| \geq 2|n| - |k|$. It follows that $\mathcal{L}$ is a bounded operator on E_α if $\gamma < \alpha < 2\gamma$. We can use the same estimate to bound the matrix elements of $\mathcal{L}$ with respect to the orthonormal basis (3.8):

$$|\langle \mathcal{L}e_{k,\alpha}, e_{n,\alpha}\rangle_\alpha| = |2w_{2n-k}\ e^{|n|\alpha}\ e^{-|k|\alpha}| \leq 2C\ e^{-(\alpha-\gamma)|k|}\ e^{-(2\gamma-\alpha)|n|}\ . \tag{3.9}$$

This implies that $\mathcal{L}$ is trace class on E_α if $\gamma < \alpha < 2\gamma$. Since $\mathcal{L}$ is trace class, it has a representation of type (2.1) with $\sum_m \lambda_m < \infty$; in fact, we have

$$\sum_n |\alpha_n| \leq \text{Tr}\ |\mathcal{L}| \leq 2C\left(\sum_{k\in\mathbf{Z}} e^{-(\alpha-\gamma)|k|}\right)\left(\sum_{n\in\mathbf{Z}}\ e^{-(2\gamma-\alpha)|n|}\right)\ . \tag{3.10}$$

We shall need this bound in order to control the rest term when we try to locate the smallest zero of the Fredholm determinant after truncation.

We are therefore in a position to apply the results of §2 to $\mathcal{L}$ on E_α; in the next section we shall see how this will then help to determine s_p. In order to be of practical use, we will need to be able to compute the Taylor coefficients of the Fredholm determinant $D_{\mathcal{L}}$ explicitly, and for this, we will need the traces Tr $\mathcal{L}^m$. As a warmup, let us compute Tr $\mathcal{L}$ itself. We have

$$\begin{aligned}\text{Tr } \mathcal{L} &= \sum_n \langle \mathcal{L} e_{n,\alpha}, e_{n,\alpha}\rangle_\alpha = \sum_n (\mathcal{L} e_{n,\alpha})_n \; e^{|n|\alpha} \\ &= \sum_n (\mathcal{L} e^{-in\,\cdot})_n = \frac{1}{2\pi}\sum_n \int_{-\pi}^{\pi} e^{in\omega}(\mathcal{L} e^{-in\,\cdot})(\omega) d\omega \; .\end{aligned} \tag{3.11}$$

To compute integrals of this type, we shall use the following standard lemma, which crops up in any study using these operators (see *e.g.*, [2], [15], [34], [18]).

Lemma 3.1 *Let w be a 2π-periodic function, and let $\mathcal{L}$ be defined as in (3.1). Then, for any $k > 0$ and any f, g 2π-periodic functions, we have*

$$\begin{aligned}\int_{-\pi}^{\pi} f(\omega)(\mathcal{L}^k g)(\omega) d\omega &= \int_{-2^k\pi}^{2^k\pi} f(\omega) \left[\prod_{\ell=1}^{k} w(2^{-\ell}\omega)\right] g(2^{-k}\omega) d\omega \\ &= 2^k \int_{-\pi}^{\pi} f(2^k\omega) \left[\prod_{m=0}^{k-1} w(2^m\omega)\right] g(\omega) d\omega \; .\end{aligned} \tag{3.12}$$

Applying this to (3.11), we find

$$\text{Tr } \mathcal{L} = \frac{1}{2\pi}\sum_n 2\int_{-\pi}^{\pi} e^{2in\omega} w(\omega) e^{-in\omega} d\omega = 2\sum_n w_n = 2w(0) \; .$$

Similarly, we can compute, for any $k \geq 1$,

$$\begin{aligned}\text{Tr } \mathcal{L}^k &= \sum_n (\mathcal{L}^k e^{-in\,\cdot})_n = \frac{1}{2\pi}\sum_n \int_{-\pi}^{\pi} e^{in\omega}(\mathcal{L}^k e^{-in\,\cdot})(\omega) d\omega \\ &= 2^k \sum_n \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(2^k-1)n\omega} \left[\prod_{m=0}^{k-1} w(2^m\omega)\right] \; d\omega = 2^k \sum_n (W_k)_{(2^k-1)n} \; ,\end{aligned} \tag{3.13}$$

where the 2π-periodic function W_k is defined by

$$W_k(\omega) = \prod_{m=0}^{k-1} w(2^m\omega) \; ,$$

and where its Fourier coefficients are denoted by $(W_k)_\ell$, as usual. Sums of the type (3.13) can be computed by means of the following lemma, which is essentially a version of the Poisson summation formula.

Lemma 3.2 *Let f be a 2π-periodic function, and f_n its Fourier coefficients. Assume that $\sum_n |f_n| < \infty$. Then, for any $\ell \geq 1$,*

$$\sum_{m=0}^{\ell-1} f\left(\frac{2\pi m}{\ell}\right) = \ell \sum_{n\in\mathbf{Z}} f_{\ell n} \ . \tag{3.14}$$

This can then be used to give an explicit formula for Tr $\mathcal{L}^k$. The following theorem summarizes the findings of this section so far.

Theorem 3.3 *Let $w(\omega)$ be a 2π-periodic function with Fourier coefficients satisfying (3.3). Define $\mathcal{L}$ to be the corresponding transfer operator, as in (3.1), and let E_α be the Hilbert spaces defined by (3.5). If $\gamma < \alpha < 2\gamma$, then $\mathcal{L}$ is a trace class operator on E_α. The spectrum of $\mathcal{L}$ does not depend on the choice of α in $]\gamma, 2\gamma[$, and for any $k \geq 1$, the traces* Tr $\mathcal{L}^k$ *are given by the explicit formula*

$$\text{Tr } \mathcal{L}^k = \frac{2^k}{2^k-1} \sum_{m=0}^{2^{k-2}} \left[\prod_{\ell=0}^{k-1} w\left(2^\ell \frac{2\pi m}{2^k-1}\right)\right] \ . \tag{3.15}$$

These results can be generalized to the multidimensional case. We replace then (3.1) by, for $\omega \in [-\pi, \pi]^d$,

$$(\mathcal{L}f)(\omega) = \sum_{j=0}^{|\det\ D|-1} w(D^{-1}\omega + \xi_j) f(D^{-1}\omega + \xi_j) \ , \tag{3.16}$$

where D is a $d \times d$ matrix with integer entries and all its eigenvalues strictly larger than 1 in absolute value, and where f, w are functions in d variables, 2π-periodic in each (*i.e.*, they are functions on the torus $\mathbb{T}^d$); the ξ_j are defined by $\xi_j = D^{-1}\zeta_j$, where the ζ_j are distinct elements in $2\pi\mathbf{Z}^d/D\mathbf{Z}^d$, so that $D^{-1}\omega + \xi_j$ are exactly the $|\det\ D|$ distinct pre-images of ω under the map D. Note that (3.16) can also be rewritten as

$$(\mathcal{L}f)_n = |\det\ D| \sum_k w_{Dn-k} f_k \ , \tag{3.17}$$

where f_k again denotes the k-th Fourier coefficient of f; the summation index k now ranges over $\mathbf{Z}^d$. As before, we shall assume

$$|w_n| \leq C\ e^{-\gamma|n|} \ , \tag{3.18}$$

with $|n| = [n_1^2 + \cdots + n_d^2]^{1/2}$; the space E_α is then defined by

$$E_\alpha = \left\{ f \text{ function on } [0, 2\pi]^d;\ \|f\|_\alpha^2 = \sum_{n\in\mathbf{Z}^d} |f_n|^2\ e^{2|n|\alpha} < \infty \right\} . \tag{3.19}$$

One then has the following generalization of Theorem 3.3:

Theorem 3.4 *Let D be a $d \times d$ matrix with integer entries and with all its eigenvalues strictly larger than 1 in absolute value. Define the spaces E_α and the operator $\mathcal{L}$ as in (3.17)–(3.19). Then $\mathcal{L}$ is trace class on E_α if $\gamma < \alpha < r_D\gamma$, where $r_D = \min\{|\lambda|;\ \lambda$ is eigenvalue of $D\} > 1$, and the spectrum of $\mathcal{L}$ on E_α does not depend on α.*

Define now, for any $k \geq 1$, the set F_k by

$$F_k = \{\eta \in\ ]-\pi,\pi[^d;\ D^k\eta - \eta \in 2\pi\mathbf{Z}^d\}\ ; \tag{3.20}$$

this set has exactly $|\det\ (D^k - \mathrm{Id})|$ elements, which are the fixed points in $\mathbb{T}^d$ of D^k. Then the traces $\mathrm{Tr}\ \mathcal{L}^k$ are given by the following explicit formula:

$$\mathrm{Tr}\ \mathcal{L}^k = \frac{|\det\ D|^k}{|\det\ (D^k - \mathrm{Id})|} \sum_{\eta\in F_k} \left[\prod_{m=0}^{k-1} w(D^m\eta)\right]\ . \tag{3.21}$$

§4 Regularity estimates using the spectral radius of $\mathcal{L}$

In this section, we discuss the relation between the Sobolev regularity of a refinable function and the spectral radius, in the previously described function spaces, of certain transfer operators that are associated to this function. More precisely, we shall see that the study of these operators leads to an exact estimate of the L^p-Sobolev exponents s_p defined in the introduction.

Let $\varphi(x)$ be an L^1 solution of the refinement equation (1.1). We assume that the coefficients h_n are summable, and that φ is normalized in the sense that $\int \varphi = 1$. By integrating on both sides of (1.1), we obtain

$$\sum_n h_n = 1\ . \tag{4.1}$$

Define the continuous function $m(\omega) = \sum_n h_n e^{-in\omega}$. In all that follows, we shall assume that $m(\omega)$ can be put in the factorized form

$$m(\omega) = \cos^N(\omega/2) q(\omega)\ , \tag{4.2}$$

where N is a strictly positive integer and $q(\omega)$ is a 2π-periodic function whose Fourier coefficients c_n satisfy a geometric decay estimate,

$$|c_n| \leq Ce^{-\beta n}\ . \tag{4.3}$$

We shall often also impose that $q(\omega)$ does not vanish on $[0, 2\pi]$.

Consequently, $m(\omega)$ is a smooth 2π-periodic function and (4.1) implies that $m(0) = 1$. By applying the Fourier transform to (1.1), one obtains

$$\hat{\varphi}(\omega) = m(\omega/2)\hat{\varphi}(\omega/2) \tag{4.4}$$

and by iteration $\hat{\varphi}(\omega)$ can be written as the pointwise convergent infinite product

$$\hat{\varphi}(\omega) = \prod_{k=1}^{+\infty} m(2^{-k}\omega) \ . \tag{4.5}$$

Note that the infinite product formula (4.5) shows that the function φ is the limit of a stationary subdivision scheme. A necessary condition for the existence of a non-trivial limit is $m(0) = 1$, *i.e.*, (4.1). Formula (4.5) only indicates, however, that this scheme converges "weakly", *i.e.*, in the sense of tempered distributions. The limit φ itself may be a tempered distribution without any regularity. More hypotheses are necessary to ensure convergence of the subdivision scheme to a continuous or more regular limit function. In particular, it has been proved by Dyn and Levin [14] that the limit function can be continuous only if $m(\pi) = 0$. This justifies the factorization of $m(\omega)$ expressed in (4.2). Additionally, strong convergence of the subdivision scheme can hold only if there exists a compact set K, congruent to $[-\pi, \pi]$ modulo 2π (*i.e.*, $|K| = 2\pi$, and for any $\omega \in [-\pi, \pi]$ there exists $\omega' \in K$ so that $\omega - \omega' \in 2\pi\mathbf{Z}$), containing a neighborhood of 0, such that $\inf_{j\geq 1, \omega\in K} |m(2^{-j}\omega)| > 0$. (See [2].) When this holds, we shall say that m is of type C.

The next lemma gives the decay of the Fourier coefficients of $|q(\omega)|^2$. Its proof is trivial.

Lemma 4.1 *The Fourier coefficients* $\{d_n\}_{n\in\mathbf{Z}}$ *of* $|q(\omega)|^2$ *satisfy the estimate*

$$|d_n| \leq C\ e^{-\beta|n|} \ . \tag{4.6}$$

From the results of the previous section, we thus know that the transfer operators associated to the functions $|q(\omega)|^2$ are trace class on E_α, for any $\alpha \in]\beta, 2\beta[$.

The operator $\mathcal{L}_{|q|^2}$ will be used to estimate the regularity of φ. In our proofs we shall use that $\mathcal{L}_{|q|^2}$ is a positive operator, in the sense that $(\mathcal{L}_{|q|^2}f)(\omega) \geq 0$ for all $\omega \in [-\pi, \pi]$ if $f(\omega) \geq 0$ for all $\omega \in [-\pi, \pi]$. Such operators have special spectral properties; see *e.g.*, [31] or [32]. To see how the general theorems on positive operators apply here, we first need to establish some facts about E_α. Define $E_\alpha^+ = \{f \in E_\alpha;\ f(\omega) \geq 0\ for\ \omega \in [-\pi, \pi]\}$. This is a cone in E_α, which contains in particular all the positive trigonometric polynomials. It follows that the closed linear span of E_α^+ equals E_α, or in the terminology of Schaeffer [31], E_α is an ordered Banach space with total positive cone. It then already follows from the Krein-Rutman theorem (see *e.g.*, [31], p. 265) that

Lemma 4.2 *The spectral radius r of $\mathcal{L}_{|q|^2}$ in E_α is an eigenvalue for $\mathcal{L}_{|q|^2}$ and there exists a positive eigenfunction for this eigenvalue.*

(The statement of the Krein-Rutman theorem in [31] is for real ordered Banach spaces, but since E_α can easily be seen to be the complexification of $\{f \in E_\alpha;\ f(\omega) \in \mathbb{R} \text{ for all } \omega \in [-\pi, \pi]\}$, the theorem still applies.)

More restrictions on the spectrum of $\mathcal{L}_{|q|^2}$ can be derived if q satisfies extra conditions. We shall need the following lemmas, the proof of which we omit here.

Lemma 4.3 *Let $w(\omega)$ be a 2π-periodic function satisfying (3.3). Assume furthermore that $w(\omega) \geq 0$ for all $\omega \in [-\pi, \pi]$, $w(0) = 1$, $w(\pi) \neq 0$, and that w is of type C. Then, for all $f \in E_\alpha$ (with $\gamma < \alpha < 2\gamma$) with $f \geq 0$ and for all $\omega \in [-\pi, \pi]$, there exists $n \geq 1$ such that $(\mathcal{L}_w^n f)(\omega) > 0$.*

Lemma 4.4 *Let m, q be as in (4.2), (4.3), with $q(0) = 1$, $q(\pi) \neq 0$. Assume moreover that m is of type C. Then r, the spectral radius of $\mathcal{L}_{|q|^2}$ on E_α, $\beta < \alpha < 2\beta$, is an eigenvalue of algebraic multiplicity 1, and the corresponding eigenfunction is strictly positive. Moreover, $r > 1$.*

The proof uses Lemma 4.3, which implies, in the terminology of Schaefer [31], that $\mathcal{L}_{|q|^2}$ is irreducible. It then follows from Theorem 3.3 in the Appendix of [31] that the algebraic multiplicity of r is 1 and that the associated eigenfunction is strictly positive; this can then be used to show that $r > 1$.

Using techniques from Hervé [20], one can also prove the following useful lemma:

Lemma 4.5 *Let m, q be as in Lemma 4.4. Then r is the only eigenvalue of $\mathcal{L}_{|q|^2}$ in the peripheral spectrum, i.e., all the other eigenvalues λ satisfy $|\lambda| < r$.*

We are now ready to state the result that links the regularity of φ with the spectral properties of transfer operators.

Theorem 4.6 *Assume that $m(\omega)$, $q(\omega)$ satisfy the same conditions as in Lemma 4.4. Let r be the spectral radius of the operator $\mathcal{L}_{|q|^2}$ on E_α, for any $\alpha \in\,]\beta, 2\beta[$. Then the Sobolev exponent s of φ satisfies*

$$s = N - \frac{1}{2}\log_2(r) \ . \tag{4.7}$$

Proof: See [5]. ■

Note that similar theorems can be proved for L^p-Sobolev exponents s_p, with $p \neq 2$, possibly under stronger conditions on m, q; see [5].

§5 Numerical precision

Combining the results of the previous sections, we immediately obtain

Theorem 5.1 *For a function φ as defined by (4.5), with $m(\omega)$, $q(\omega)$ satisfying the conditions in Lemma 4.4, the Sobolev exponent of φ can be expressed as $s = N - \frac{1}{2}\log_2(r)$, where $(r)^{-1} = x$ is the zero of smallest absolute value of the Fredholm determinant $d(z)$ of the operator $\mathcal{L}_{|q|^2}$ associated to the weight function $w(\omega) = |q(\omega)|^2$.*

Formulas (2.9), (2.13), and (3.15) give us an explicit expression for the Taylor series of the analytic function $d(z)$. In practice, to estimate x numerically, we are obliged to truncate this series and work with the polynomials that are obtained from the first order terms. In order to have an idea how precise our estimate is for x after this truncation, we need to measure how well x is approximated by the smallest zero of the truncated series at a given order. In Cohen and Daubechies [5], such precise estimates are obtained by applying Rouché's theorem. Other techniques can be used as well. In all cases, it is crucial that we can control the tail $R_A^N(z)$ as in (2.12).

Note that because of Lemma 4.5, one can write a simpler formula for r, namely

$$r = \lim_{n\to\infty} |\ \mathrm{Tr}\ \mathcal{L}^n_{|q|^2}|^{1/n}\ .$$

This avoids the search for zeros of the Fredholm determinant altogether. In this case, however, one has no control over the error, and because $\mathcal{L}_{|q|^2}$ is a non-selfadjoint operator, with complex eigenvalues, the behavior of $|\ \mathrm{Tr}\ \mathcal{L}^n_{|q|^2}|^{1/n}$, as a function of n, can be very deceptive. See [5] for a more extensive discussion, with examples.

§6 Applications

All our examples are motivated by wavelet constructions; we take the refinable function φ to be either the orthonormal scaling function in a multiresolution analysis, or the autocorrelation function of a scaling function. We start by recalling some pertinent definitions.

The refinable function $\varphi(x)$ is said to be *cardinal interpolant* if it satisfies the condition

$$\varphi(k) = \delta_{0,k}, \quad k \in \mathbf{Z}\ ; \tag{6.1}$$

it is called *orthonormal* if

$$\int \varphi(x-k)\ \overline{\varphi(x-\ell)}\ dx = \delta_{k,\ell}, \quad k,\ell \in \mathbf{Z}\ . \tag{6.2}$$

These properties correspond to special constraints on $m(\omega)$: (6.1) implies

$$m(\omega) + m(\omega + \pi) = 1 \ , \tag{6.3}$$

whereas (6.2) can hold only if

$$|m(\omega)|^2 + |m(\omega + \pi)|^2 = 1 \ . \tag{6.4}$$

The conditions (6.3) or (6.4) are necessary for (6.1) or (6.2) to hold, but not sufficient. Under additional technical conditions that ensure uniform convergence of the subdivision algorithm in the first case, or L^2-convergence in the second case, (6.3) implies (6.1) and (6.4) implies (6.2). (For a detailed discussion, see Chapter 6 in [8].)

It is clear that cardinal interpolation and orthonormality are linked: if ϕ is an orthonormal refinable function, then its autocorrelation function $\Phi(x) = \int \phi(y) \ \overline{\phi(x-y)} \ dy$ is interpolating; the corresponding functions m_ϕ and m_Φ are related by $m_\Phi = |m_\phi|^2$. In fact, compactly supported wavelets are usually constructed by first identifying a suitable positive m_Φ and then constructing m_ϕ so that $|m_\phi|^2 = m_\Phi$. It is then obvious that the L^p-Sobolev exponents of ϕ and Φ are related by

$$s_p(\Phi) = 2 s_{2p}(\phi) \ . \tag{6.5}$$

In particular, using the definitions (1.2) and (1.3), we find

$$\mu(\Phi) = s_1(\Phi) = 2 s_2(\phi) = 2 s(\phi) \ , \tag{6.6}$$

where the first equality is a consequence of $\hat{\Phi}(\omega) \geq 0$ for all $\omega \in \mathbb{R}$.

In the first subsection below we concentrate on families of examples where m_Φ is a positive trigonometric polynomial of the form $P_\Phi(\cos\omega)$, so that Φ is real, symmetric and compactly supported. By Riesz' spectral factorization lemma, we can then find a trigonometric polynomial m_ϕ, with real coefficients, so that $|m_\phi|^2 = m_\Phi$. The corresponding refinable functions ϕ are then compactly supported scaling functions from which compactly supported wavelets can be constructed; see [8]. In the second subsection we consider examples where $P_\Phi(\cos\omega)$ is no longer a trigonometric polynomial, but the quotient of two such polynomials. The third subsection of examples looks at some two-dimensional examples $m(\omega_1, \omega_2)$ which cannot be written as products $m_1(\omega_1) m_2(\omega_2)$ of one-dimensional functions, with matrix dilations. Finally, in the fourth subsection we use the results of the previous subsections to deal with a problem on spline wavelet bases.

6.1 Interpolating and orthonormal scaling functions with compact support

The minimal degree solution to (6.3) and the factorization requirement (4.2) is given by

$$m_N(\omega) = \left(\cos\frac{\omega}{2}\right)^{2N} \sum_{j=0}^{N-1} \binom{N-1+j}{j} \left(\sin\frac{\omega}{2}\right)^{2j} .$$

(We are interested in factoring out only even powers of $\cos\frac{\omega}{2}$ because we want $m_N(\omega) = P_N(\cos\omega)$.) In this case $q(\omega)$ is clearly strictly positive for all $\omega \in [0, 2\pi]$, so that we can apply our theorems for all values $p \geq 1$. The corresponding functions Φ_N, ϕ_N have been studied extensively (see *e.g.*, [8] for many references). We have computed the L^p-Sobolev exponents of these functions for different values of N. Table 1 shows the $s_p(\phi_N)$ for $p = 1, 2, 4, 8$, $N = 1, \ldots, 19$.

Table 1.

L^p-Sobolev exponents of φ_N, $p = 1, 2, 4, 8$, $N = 1, \ldots, 19$ (Polynomial).

$N \backslash p$	1	2	4	8
1	−0.322289	0.338856	0.669428	0.834714
2	0.521293	0.999820	1.220150	1.310014
3	0.979675	1.414947	1.587361	1.631686
4	1.391644	1.775305	1.896446	1.912144
5	1.767934	2.096541	2.171522	2.174682
6	2.116733	2.388060	2.430780	2.431755
7	2.441544	2.658569	2.680780	2.680307
8	2.746639	2.914556	2.926425	2.925926
9	3.035292	3.161380	3.166924	3.165533
10	3.309107	3.402546	3.405193	3.405141
11	3.572141	3.639569	3.641221	3.638529
12	3.825525	3.873991	3.874236	3.871917
13	4.071021	4.105802	4.105736	4.105305
14	4.311641	4.336042	4.336476	4.335502
15	4.547368	4.564708	4.564925	4.562449
16	4.780028	4.792323	4.792608	4.792645
17	5.010231	5.018884	5.018754	5.016283
18	5.238588	5.244390	5.244127	5.243230
19	5.464480	5.468841	5.468728	5.466868

An interesting observation is that $s_p(\varphi_N)$ becomes independent of p as N goes to $+\infty$. This reflects the fact that φ_N has a lacunary structure in the Fourier domain and that this phenomenon grows with n. More precisely, if the Fourier transform of a function φ decays uniformly at infinity in the sense that $C_1(1+|\omega|)^{-\alpha} \le |\hat{\varphi}(\omega)| \le C_2(1+|\omega|)^{-\alpha}$, then the exponents $s_p(\varphi) = \alpha - \frac{1}{p}$ are related by $s_p - s_q = \frac{1}{q} - \frac{1}{p}$. This is true here only for $N = 1$ (which corresponds to the box function $\varphi_1(x) = \chi_{[0,1]}(x)$); $\hat{\varphi}_1(\omega) = \frac{1-e^{-i\omega}}{i\omega}$ decays uniformly in $|\omega|^{-1}$, up to the oscillation of the numerator. For larger N, the lacunary structure takes over. As shown in Volkmer [36] and Cohen and Conze [3], the worst decay occurs at the points $\omega_j = 2^j\left(\frac{2\pi}{3}\right)$, $j > 0$. A possible explanation for our observation could be that the L^p norm of $\hat{\varphi}_N$ concentrates at these points, as N grows.

We thus conjecture that for all $p, q > 0$, $\lim_{N\to+\infty} |s_p(\varphi_N) - s_q(\varphi_N)| = 0$. If this is true, then we have in particular $\lim_{N\to+\infty} |s(\varphi_N) - \mu(\varphi_N)| = 0$ since $s_1(\varphi_N) \le \mu(\varphi_N) \le s(\varphi_N) = s_2(\varphi_N)$.

For large values of N, our method allows us to observe the asymptotic behavior of $\mu(\varphi_N)$.

It was proved by Volkmer [36] that $\lim_{N\to+\infty} \frac{\mu(\varphi_N)}{N} = \lim_{N\to+\infty} \frac{s(\varphi_N)}{N} = 1 - \frac{\log_2 3}{2} \simeq 0.2075$. The values of $s_2(\varphi_N) = s_2(N)$ presented in Table 1 show in addition that $s_2(N) - \left(1 - \frac{\log_2 3}{2}\right) N$ stays bounded by 3 for $N \le 100$.

6.2 Interpolating and orthonormal scaling functions with infinite support

We now turn to the solutions of (6.3) that have the factorized form (4.2) but are not necessarily trigonometric polynomials. We shall look for solutions of the type

$$m(\omega) = \cos^{2N}\left(\frac{\omega}{2}\right)\ R(\cos\omega)\ , \qquad (6.7)$$

where $R(z) = \frac{P(z)}{Q(z)}$ is a rational function that is strictly positive on $[-1, 1]$. Under this hypothesis we know that we can apply our method to estimate the L^p-Sobolev exponent s_p of the associated scaling function since the Fourier coefficients of $|R(z)|^p$ have exponential decay. Note that the scaling function φ is not compactly supported but typically still has exponential decay at infinity (some restrictions on R, always satisfied in practical examples, are needed to ensure this).

The choice of a rational function is still useful in the applications where one has to perform discrete convolutions with the Fourier coefficients of $m(\omega)$: although they are not finite in number, these convolutions can be implemented in a fast recursive way, the complexity being roughly $2S \times [\deg(P) + \deg(Q) + N]$ where S is the size of the input data.

The simplest rational solution of (6.3) of the form (6.7) is given by the family

$$m_N(\omega) = \cos^{2N}\left(\frac{\omega}{2}\right) R_N(\cos\omega) \qquad (6.8)$$

with $R_N(\cos\omega) = \left[\cos^{2N}\left(\frac{\omega}{2}\right) + \sin^{2N}\left(\frac{\omega}{2}\right)\right]^{-1}$. These solutions are well known in signal processing as the transfer functions of the so-called "Butterworth filters" (see [27] for a detailed review).

As in the previous section, we give the estimate of s_p for the orthonormal scaling functions φ_N, $1 \le N < 20$ and $p = 1, 2, 4, 8$. It is interesting to see that these exponents remain substantially different as N grows: the lacunary behavior does not prevail as much as in the compactly supported case.

For large values of N, we have examined the evolution of $s(\varphi_N) = s(N)$ (see Table 2). It reveals a linear asymptotic behavior, similar to the compactly supported case.

Note that the limit ratio $\frac{s(N)}{N} \simeq .8$ seems to indicate that the worst decay of $\hat{\varphi}_N(\omega)$ occurs at the points $\omega_j = 2^j\left(\frac{2\pi}{3}\right)$. Indeed, we have

$$|\hat{\varphi}_N(\omega_j)| = \left|\hat{\varphi}_N\left(\frac{\pi}{3}\right)\right| \cdot \left|m_N\left(\frac{2\pi}{3}\right)\right|^{j/2} = C|\omega_j|^{r_N} \tag{6.9}$$

with $r_N = \frac{1}{2}\log_2\left(R_N\left(\frac{1}{2}\right)\right) - 1$. From the definition of R_N, we obtain

$$\lim_{N\to+\infty} \frac{r_N}{N} = -\frac{1}{2}\log_2 3 \simeq -0.7925 \ , \tag{6.10}$$

which seems to coincide with the experimental asymptotic ratio.

The Butterworth functions $R_N(\cos\omega)$ correspond to a choice with $P(z) = 1$, $R(z) = \frac{1}{Q(z)}$, which makes them in some sense opposites to the polynomial solutions of the previous subsection, for which $Q(z) = 1$, $R(z) = P(z)$. Recently, intermediate solutions that are equally balanced between the numerator and the denominator were proposed by Herley and Vetterli [19]. Cohen and Daubechies [5] contains tables for their $s_p(N)$ as a function of N.

Table 2.

L^p-Sobolev exponents of φ_N, $p = 1, 2, 4, 8$, $N = 1, \ldots, 19$ (Butterworth).

$N\backslash p$	1	2	4	8
1	−0.322289	0.338856	0.669428	0.834714
2	0.677350	1.256211	1.495117	1.604344
3	1.561362	2.044109	2.269688	2.365870
4	2.370365	2.843768	3.059757	3.148599
5	3.183890	3.648646	3.857332	3.940563
6	3.999055	4.456118	4.658210	4.735925
7	4.815040	5.264533	5.460184	5.532265
8	5.630616	6.072947	6.262157	6.328326
9	6.446191	6.881125	7.063818	7.123827
10	7.260947	7.688598	7.864696	7.918627
11	8.075292	8.495600	8.664948	8.712863
12	8.888817	9.301894	9.464414	9.506534
13	9.701520	10.107480	10.263251	10.299921
14	10.513813	10.912358	11.061142	11.093166
15	11.325284	11.716526	11.858558	11.885986
16	12.135933	12.519984	12.655184	12.678805
17	12.946170	13.322968	13.451333	13.471625
18	13.755996	14.125241	14.247006	14.264159
19	14.564999	14.927039	15.042202	15.056836

6.3 Nonseparable bidimensional scaling functions

The simplest way to generate multivariate scaling functions is to use the tensor product, *i.e.*, to define

$$\Phi(x_1, \ldots, x_n) = \varphi_1(x_1) \cdots \varphi_n(x_n) \ , \tag{6.11}$$

where $\varphi_1, \ldots, \varphi_n$ are univariate refinable functions. Note that if the univariate functions are cardinal interpolant or orthonormal, then the same property holds for Φ. The analysis of the regularity of Φ then follows directly from the univariate analysis on the φ_j's.

One of the simplest—yet instructive—situations where non-separable scaling functions are unavoidable corresponds to the choice

$$D = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{6.12}$$

for the dilation matrix. In that case the interpolatory condition has the following formulation:

$$M(\omega_1, \omega_2) + M(\omega_1 + \pi, \omega_2 + \pi) = 1 \ . \tag{6.13}$$

One can use the univariate functions $m_N(\omega)$, defined in the first subsection, to derive a solution of (6.13) as follows:

$$M_N(\omega_1,\omega_2) = [c(\omega_1,\omega_2)]^N \sum_{j=0}^{N-1} \binom{N-1+j}{j} [s(\omega_1,\omega_2)]^j \;, \tag{6.14}$$

where

$$c(\omega_1,\omega_2) = \frac{1}{2}\left(\cos^2\left(\frac{\omega_1}{2}\right) + \cos^2\left(\frac{\omega_2}{2}\right)\right)$$
$$\text{and } s(\omega_1,\omega_2) = \frac{1}{2}\left(\sin^2\left(\frac{\omega_1}{2}\right) + \sin^2\left(\frac{\omega_2}{2}\right)\right) \;.$$

We denote by Φ_N the nonseparable cardinal interpolant functions associated to M_N. Note however that the Riesz factorization lemma does not generalize in n dimensions, $n > 1$, so that it is not possible to derive compactly supported orthonormal scaling functions from these Φ_N. Using the preliminary results of example 3.8, we can compute the Hölder exponents $\mu(\Phi_N) = s_1(\Phi_N)$ as well as $s_p(\Phi_N)$ for $p = 2, 4, 8$. We display their values, for $N = 1, \ldots, 19$ in Table 3 below. As in the univariate case, this table reveals the increasingly lacunary structure of the functions Φ_N in the Fourier domain.

Table 3.

L^p-Sobolev exponent of Φ_n, $p = 1, 2, 4, 8$, $N = 1, \ldots, 19$.

$N \backslash p$	1	2	4	8
1	0.611268	1.575915	1.939386	1.981617
2	2.285413	3.249338	3.684182	3.862247
3	3.881443	4.778977	5.146044	5.274342
4	5.395644	6.199651	6.487579	6.582461
5	6.841235	7.549708	7.780608	7.868502
6	8.233367	8.854675	9.054724	9.141341
7	9.584611	10.132740	10.318169	10.404515
8	10.904434	11.395230	11.573954	11.660232
9	12.200067	12.648141	12.823734	12.909992
10	13.477248	13.894538	14.068620	14.154872
11	14.740529	15.136095	15.309421	15.395672
12	15.993417	16.373816	16.546755	16.633005
13	17.238515	17.608370	17.781105	17.867355
14	18.477713	18.840232	19.012859	19.099109
15	19.712362	20.069763	20.242333	20.328582
16	20.943434	21.297249	21.469787	21.556036
17	22.171630	22.522919	22.695440	22.781690
18	23.397462	23.746966	23.919477	24.005727
19	24.621314	24.969550	25.142055	25.228305

6.4 A problem on spline wavelet bases

Spline wavelets are generated by a function ψ that is piecewise polynomial (of a fixed degree d) on each integral $[k/2,(k+1)/2[$, $k \in \mathbf{Z}$. Such a wavelet is given by

$$\psi(x) = \sum_{n\in\mathbf{Z}} g_n \varphi(2n-n) \tag{6.15}$$

where $\varphi = \chi_{[0,1]} * \chi_{[0,1]} \cdots * \chi_{[0,1]}$ ($d+1$ times) is the box spline of degree d, and g_n is an oscillating ℓ^2 sequence, *i.e.*, $\sum_n g_n = 0$. All the examples of known spline wavelet bases (orthonormal, semi-orthonormal, biorthogonal) are of this form, with particular g_n.

One can then ask the following question: given an arbitrary oscillating sequence g_n, when does the corresponding combination (6.15) of box-splines generate a (Riesz) wavelet basis $\{\psi_{j,k}\}_{j,k\in\mathbf{Z}}$? In particular, we have in mind very simple sequences such as $g_0 = 1$, $g_1 = -1$ or $g_0 = -1$, $g_1 = 2$, $g_2 = -1$, etc.

First, note that ψ can be written in the Fourier domain as

$$\hat{\psi}(\omega) = m_1\left(\frac{\omega}{2}\right)\hat{\varphi}\left(\frac{\omega}{2}\right) , \tag{6.16}$$

where $m_1(\omega) = \sum_n g_n e^{-in\omega}$. Moreover, we have

$$\hat{\varphi}(\omega) = \prod_{k=1}^{+\infty} m_0(2^{-k}\omega) , \tag{6.17}$$

where $m_0(\omega) = \left(\frac{1+e^{-i\omega}}{2}\right)^d$. In the case of biorthogonal wavelets, $m_1(\omega)$ is equal to $e^{-i\omega}\overline{\tilde{m}_0(\omega+\pi)}$, where $\tilde{m}_0$ generates the dual scaling function $\tilde{\varphi}$ in the sense that

$$\hat{\tilde{\varphi}}(\omega) = \prod_{k=1}^{+\infty} \tilde{m}_0(2^{-k}\omega) . \tag{6.18}$$

The biorthogonality constraint is expressed by the equation

$$\overline{m_0(\omega)}\ \tilde{m}_0(\omega) + \overline{m_0(\omega+\pi)}\ \tilde{m}_0(\omega+\pi) = 1 , \tag{6.19}$$

or, equivalently,

$$e^{i\omega}[m_0(\omega)m_1(\omega+\pi) - m_0(\omega+\pi)m_1(\omega)] = 1 . \tag{6.20}$$

It is clear that equation (6.20) is a strong restriction on m_1. Given a solution of (6.20), one can however construct other m_1 that still give rise to Riesz bases $\psi_{j,k}$. It suffices to take $m_1(\omega) = m(2\omega)M_1(\omega)$ where $M_1(\omega)$ satisfies equation (6.20) and $m(\omega)$ is a 2π-periodic function such that

$$0 < c \leq |m(\omega)| \leq C < \infty \quad \text{a.e.} \;\; \omega \in \mathbb{R} \; . \tag{6.21}$$

This corresponds to the choice

$$\hat{\psi}(\omega) = m(\omega) M_1 \left(\frac{\omega}{2}\right) \hat{\varphi} \left(\frac{\omega}{2}\right) \; . \tag{6.22}$$

We can then define Ψ by $\hat{\Psi}(\omega) = M_1\left(\frac{\omega}{2}\right)\hat{\varphi}\left(\frac{\omega}{2}\right)$, and use the biorthogonal theory to study if $\{\Psi_{j,k}\}_{j,k\in\mathbf{Z}}$ is a Riesz basis of $L^2(\mathbb{R})$. If this is the case, then the same clearly holds for $\{\psi_{j,k}\}_{j,k\in\mathbf{Z}}$. Note that $m(\omega)$ and $M_1(\omega)$ are completely determined from $m_1(\omega)$ since we have $M_1(\omega) = m_1(\omega)/m(2\omega)$ and thus, by (6.20),

$$m(2\omega) = e^{i\omega}[m_0(\omega)m_1(\omega+\pi) - m_0(\omega+\pi)m_1(\omega)] \; . \tag{6.23}$$

The system $\{\psi_{j,k}\}_{j,k\in\mathbf{Z}}$ will thus constitute a Riesz basis if the two following conditions are satisfied:

(i) the function $m(\omega)$ defined by (6.23) is bounded below and above by strictly positive constants.

(ii) $\{\Psi_{j,k}\}_{j,k\in\mathbf{Z}}$ is a Riesz basis. A necessary and sufficient condition for this to hold was given in [4]. In our context, this results in the following

Theorem 6.1 *Let* $\tilde{M}_0(\omega) = -e^{-i\omega}\overline{M_1(\omega+\pi)}$ *and* $M(\omega) = |\tilde{M}_0(\omega)|^2/\cos^2\left(\frac{\omega}{2}\right)$. *Assume that the Fourier coefficients of* M *satisfy the decay condition (3.3). Define* $\mathcal{L}$ *as the transition operator associated to* $M(\omega)$ *and denote by* ρ *its spectral radius on a space* E_α *for any* $\alpha \in]\gamma, 2\gamma[$. *Then* $\{\Psi_{j,k}\}_{j,k\in\mathbf{Z}}$ *constitutes a Riesz basis of* $L^2(\mathbb{R})$ *if and only if* $\rho < 4$.

Note that, according to our results, this condition means that the L^2-Sobolev exponent of the scaling function $\tilde{\Phi}$ associated to $\tilde{M}_0$ is strictly positive.

Since we have

$$|\tilde{M}_0(\omega)|^2 = \frac{|m_1(\omega+\pi)|^2}{|m(2\omega)|^2} \; , \tag{6.24}$$

the function $M(\omega)$ is not a trigonometric polynomial in general, even when $\{g_n\}$ is a finite sequence. This made this application inaccessible to earlier methods that could only deal with finite masks.

An immediate application concerns the case of linear splines, *i.e.*, $\varphi(x) = \sup\{0, 1-|x|\}$. In that case, we can propose three simple wavelets corresponding to different choices for the g_n coefficients:

- $\psi_a(x) = \varphi(2x) - \varphi(2x-1)$
- $\psi_b(x) = 2\varphi(2x) - \varphi(2x-1) - \varphi(2x+1)$

- $\psi_c(x) = 2\varphi(2x-1) - \varphi(2x) - \varphi(2x-2)$

One can easily check that for ψ_b, the associated function $m(\omega)$ vanishes at some point. It is also easy to check that this implies that Span $\{\psi_b(x-k)\}_{k\in\mathbf{Z}}$ cannot complement in a stable manner the space V_0 into V_1.

In the cases of ψ_a and ψ_c, the associated function $m(\omega)$ does not vanish, so that we can further investigate the associated functions $M(\omega)$.

For ψ_a, we find

$$M(\omega) = \frac{16}{10 + 6\cos\omega} \, , \tag{6.25}$$

and our method shows that $\rho > 4$. For ψ_c we find

$$M(\omega) = \cos^2\left(\frac{\omega}{2}\right)\left(\frac{4}{3+\cos\omega}\right)^2 \, , \tag{6.26}$$

and in that case $\rho < 4$.

Conclusion: From the three functions above, only ψ_c generates a Riesz wavelet basis.

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