Math 4108 Homework 10

Due at the beginning of class on Tuesday, March 24.

§16.10 #10
§16.11 #5

Not from the text:

1. Let M be an $n \times n$ matrix with coefficients in $\mathbf{C}$. Use the Jordan normal form of M to prove that if $M^n = I_n$, then M is diagonalizable.

2. In this problem you will prove the following generalization of Theorem 16.11.1:

   Theorem (Kummer)**.** *Let n be a positive integer and let F be a subfield of $\mathbf{C}$ containing $\boldsymbol{\mu}_n$, the nth roots of unity. Let K/F be a Galois extension with cyclic Galois group G of order n. Then $K = F(\sqrt[n]{\beta})$ for some $\beta \in F$.*

   (a) Let σ generate G. Use the previous problem to show that the eigenvalues of σ (considered as an F-linear transformation from K to itself) are nth roots of unity which generate the cyclic group $\boldsymbol{\mu}_n$.

   (b) Let λ be an eigenvalue of σ, and let $\beta \in K^\times$ be an eigenvector. Let r be the order of λ (as an element of $\boldsymbol{\mu}_n$). Prove that $\beta^r \in F$ and that $x^r - \beta^r$ is the minimal polynomial of β.

   (c) Let $\lambda_1, \ldots, \lambda_m$ be eigenvalues of σ with eigenvectors $\beta_1, \ldots, \beta_m \in K^\times$. Let $\lambda = \lambda_1 \cdots \lambda_m$ and $\beta = \beta_1 \cdots \beta_m$. Let r be the order of λ. Prove that $\beta^r \in F$ and that $x^r - \beta^r$ is the minimal polynomial of β.

   (d) Complete the proof of the Theorem.

Note:

1. To clarify #16.10.10: the index i runs from 1 to $p-1$, inclusive. You may assume the characteristic is zero. Hint: what is the sum of the γ_i?