**Math 4108 Homework 11**

Due at the beginning of class on Tuesday, March 31.

§12.2 #2, 6, 7

**Not from the text**:

1. (30 points) Let $\delta = \sqrt{-5}$ and let $\mathbf{Z}[\delta] = \{a + b\delta \ : \ a, b \in \mathbf{Z}\}$, a subring of $\mathbf{C}$.

   (a) For $z \in \mathbf{Z}[\delta]$, show that $|z|^2 \in \mathbf{Z}$, where $|z|^2$ is the usual complex absolute value.

   (b) Prove that the following sets are equal:

   $$\mathbf{Z}[\delta]^\times = \big\{z \in \mathbf{Z}[\delta] \ : \ |z|^2 = 1\big\} = \{\pm 1\}.$$

   (c) Prove that $2, 3, 1 + \delta, 1 - \delta$ are irreducible elements of $\mathbf{Z}[\delta]$, no two of which are associate. [Hint: if $2 = uv$ is a proper factorization, what is $|u|^2$?]

   (d) Use (3) and the equation

   $$2 \cdot 3 = 6 = (1 + \delta)(1 - \delta)$$

   to prove that $2, 3, 1 + \delta, 1 - \delta$ are not prime elements. Conclude that $\mathbf{Z}[\delta]$ is not a unique factorization domain.

   (e) Let $I = (2, 1 + \delta)$. Show that $I$ is generated as an *abelian group* by $2$ and $1 + \delta$, draw a picture of $I$ in the complex plane, and prove that $I$ is not the unit ideal.

   (f) Show that the ideal $I = (2, 1 + \delta)$ is not principal. [Hint: if $I = (z)$ show that $z \mid 2$ and $z \mid 1 + \delta$.]

   (g) Prove that a greatest common divisor of $3(1 + \delta)$ and $6$ does not exist in $\mathbf{Z}[\delta]$. [Hint: if a greatest common divisor $d$ exists, show that $3 \mid d$ and that $d/3$ divides $1 + \delta$ and $2$.]