Math 4803/8803 Homework 3

Due at the beginning of class on Wednesday, September 9.

Exercises in Samuel:
Chapter II #2.

Exercises not from the text:

(1) Find the Smith normal form of the following matrices.

(a) $\begin{bmatrix} 8 & 20 \\ 10 & 24 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 7 & 4 \\ 10 & 17 & 8 \end{bmatrix}$

(c) $\begin{bmatrix} 12 & 6 & 21 \\ 15 & 9 & 27 \\ 6 & 12 & 15 \end{bmatrix}$ (d) $\begin{bmatrix} 1+3x^2 & 3x^2 \\ 2x+3x^3 & x+3x^3 \end{bmatrix}$

In (d) the coefficient ring is $\mathbf{Q}[x]$.

(2) Let G be the abelian group given by the generators and relations

$$G = \langle x, y, z \mid 12x + 15y + 6z = 6x + 9y + 12z = 21x + 27y + 15z = 0\rangle.$$

Express G as a product of cyclic groups as in the classification theorem for finitely generated abelian groups, and find generators for the cyclic factors in terms of x, y, z. [Keep track of the row and column operations.]

(3) Let A be an integral domain and M an A-module. The *torsion submodule* of M is the subset of all torsion elements:

$$M_{\text{tors}} := \{x \in M \mid \exists a \in A \setminus \{0\} \text{ such that } ax = 0\}.$$

(a) Prove that M_{tors} is a submodule of M.
(b) Prove that M/M_{tors} is torsionfree.
(c) If A is a PID and $M \cong A/(a_1) \times \cdots \times A/(a_n) \times A^r$ as in the classification theorem, what are M_{tors} and M/M_{tors}? What is the annihilator of M?

(4) Let K be a field and $A = K[t]$.

(a) Show that an A-module is naturally the same thing as a K-vector space V equipped with a K-linear transformation $T\colon V \to V$. [The abelian group underlying the A-module corresponding to (V, T) is just V, with $t \cdot v = Tv$ for all $v \in V$.]
(b) Let M be an $n \times n$ matrix with coefficients in K and let $p(t)$ be the characteristic polynomial of M. Prove the Cayley–Hamilton theorem that $p(M) = 0$ in the following way. By (a), we can regard (K^n, M) as an A-module. Now follow the proof of Theorem 2.1.1(c $\Longrightarrow$ a), using the action of the determinant of $tI_n - M$.