Math 4803/8803 Homework 4

Due at the beginning of class on Wednesday, September 16.

Exercises in Samuel:
Chapter II #4.

Exercises not from the text:

(1) Let A, B, C be rings, with A a subring of B and B a subring of C.
   (a) Suppose that B is a finitely generated A-module and C is a finitely generated B-module. Prove that C is a finitely generated A-module.
   (b) Suppose that B is a finitely generated A-algebra and C is a finitely generated B-algebra. Prove that C is a finitely generated A-algebra.
   (c) Find an example of a ring which is a finitely generated $\mathbf{Q}$-algebra but not a finitely generated $\mathbf{Q}$-module.

(2) Let $\zeta = e^{2\pi i/5}$. Use the proof of Theorem II.1(c $\Longrightarrow$ a) to find an explicit equation of integral dependence for $\zeta + \zeta^2$ over $\mathbf{Z}$.

(3) Let A be a subring of B, with B integral over A. Prove that $B^\times \cap A = A^\times$. Show this is false in general without the integrality hypothesis.

(4) Let A be a subring of B, such that the set $B \backslash A$ is closed under multiplication. Show that A is integrally closed in B.

(5) Let A be a ring, let G be a finite group of automorphisms[1] of A, and let $A^G = \{x \in A \,:\, \forall \sigma \in G,\ \sigma(x) = x\}$. Prove that A is integral over A^G. [Hint: if A is a field, this is a basic fact from Galois theory — how is it proved in that context?]

[1] An *automorphism* of A is an isomorphism from A to itself. The set of all automorphisms forms a group under composition.