Math 4803/8803 Homework 5

Due at the beginning of class on Wednesday, September 23.

(1) Let x be a cube root of 2 and let $K = \mathbf{Q}(x)$. This is a cubic extension of $\mathbf{Q}$. In this problem you will show that the subring $\mathbf{Z}[x] \subset K$ is equal to the integral closure $\mathscr{O}_K$ of $\mathbf{Z}$ in K.

(a) Let $z = a + bx + cx^2 \in K$ for $a, b, c \in \mathbf{Q}$. Find the conjugates of z in $\mathbf{C}$. Calculate the trace, norm, and characteristic polynomial of z over $\mathbf{Q}$.

(b) Show that $\mathbf{Z}[x] \subset \mathscr{O}_K$, find a basis for $\mathbf{Z}[x]$ as a $\mathbf{Z}$-module, and show that $\mathrm{Frac}(\mathbf{Z}[x]) = K$.

(c) Show that $6\mathscr{O}_K \subset \mathbf{Z}[x]$. [Let $z = a + bx + cx^2 \in \mathscr{O}_K$ for $a, b, c \in \mathbf{Q}$. Calculate the traces of $z, xz,$ and x^2z.]

(d) **(Optional)** By (c) we know that if $z = a+bx+cx^2 \in \mathscr{O}_K$ then $6a, 6b, 6c \in \mathbf{Z}$. By calculating traces, norms, and characteristic polynomials, prove directly that $6a, 6b, 6c$ are multiples of 6, so that $z \in \mathbf{Z}[x]$.

(2) With the notation in (1), let e_1, e_2, e_3 be the basis for $\mathbf{Z}[x]$ that you found in (b). Let M be the matrix $(\mathrm{Tr}(e_ie_j))_{i,j=1}^3$. Calculate $\det(M)$, and show that $\det(M)$ is independent of the choice of basis. This is by definition the *discriminant* of K.

(3) Let K be a perfect field, L/K a finite extension, and C/K an algebraically closed field extension. Prove that L/K is Galois if and only if all of the K-embeddings $L \to C$ have the same image.[1]

(4) Let $K \subset L \subset L'$ be a tower of finite field extensions, let $d = [L' : L]$, and let $x \in L$. Let $p_{L'/K}(X)$ (resp. $p_{L/K}(X)$) be the characteristic polynomial of x with respect to the extension L'/K (resp. L/K). Prove that $p_{L'/K}(X) = p_{L/K}(X)^d$, and conclude that

$$N_{L'/K}(x) = N_{L/K}(x)^d \quad \text{and} \quad \mathrm{Tr}_{L'/K}(x) = d\,\mathrm{Tr}_{L/K}(x).$$

[1] Two subfields of C which are K-isomorphic are also called *conjugate*.