Math 4803/8803 Homework 6

Due at the beginning of class on Wednesday, September 30.

(1) Let K be a perfect field and let L/K and F/L be finite field extensions (so L and F are also perfect fields). Prove that

$$\mathrm{Tr}_{F/K} = \mathrm{Tr}_{L/K} \circ \mathrm{Tr}_{F/L} \quad \text{and} \quad \mathrm{N}_{F/K} = \mathrm{N}_{L/K} \circ \mathrm{N}_{F/L}\,.$$

[In class we proved that the trace and norm of an element x can be calculated in terms of the images of x under the distinct homomorphisms into an algebraically closed field extension C/K. Use that result and its proof, and be careful about what field is the ground field for which homomorphism!]

(2) Let A be a subring of a ring C such that $C \cong A^n$ as A-modules, and let $B \subset C$ be a subring containing A, so $A \subset B \subset C$. Suppose that A is a principal ideal domain.

(a) Show that B is a finitely generated free A-module.

(b) Suppose that B also has rank n. Let $x_1, \ldots, x_n \in B$ and $y_1, \ldots, y_n \in C$ be bases. Prove that

$$D(x_1, \ldots, x_n) = D(y_1, \ldots, y_n) \cdot d^2$$

for some $d \in A$.

(c) Conclude that if $D(x_1, \ldots, x_n)$ is nonzero and squarefree then $B = C$.

(3) Let $f(X) = X^3 + 2X + 1 \in \mathbf{Q}[X]$, let $x \in \mathbf{C}$ be a root of f, and let $K = \mathbf{Q}(x)$.

(a) Prove that f is irreducible over $\mathbf{Q}$.

(b) Let $B = \mathbf{Z}[x]$. Show that B is a free $\mathbf{Z}$-module with basis $1, x, x^2$, and calculate $D(1, x, x^2)$.

(c) Use (b) and problem (2) to prove that B is the ring of integers of K.

(d) Regarding $1, x, x^2$ as a basis of $K/\mathbf{Q}$, compute the dual basis of K with respect to the trace pairing. (This is the inverse image of the dual basis of K^* under the isomorphism $K \xrightarrow{\sim} K^*$ given by the trace pairing.)

If you're not yet convinced that discriminants are useful, try proving directly that B is integrally closed!

(4) Let d, d' be squarefree integers, let $K = \mathbf{Q}(\sqrt{d}, \sqrt{d'})$, and let $\mathscr{O}_K$ be the ring of integers of K.

(a) Let $a \in \mathbf{Z}$ be a generator of $\mathscr{D}_{\mathscr{O}_K/\mathbf{Z}}$, and suppose that p is a prime factor of a. Prove that $p \mid 2dd'$. [First compute the discriminant of $\mathbf{Z}[\sqrt{d}, \sqrt{d'}]$.]

(b) Let p, q, ℓ be distinct primes, with $\ell \neq 2$. Use (a) to prove that ℓ is not a square in $\mathbf{Q}(\sqrt{p}, \sqrt{q})$.

(c) **(Optional)** Prove a statement like (a) for three squarefree integers d, d', d''. Can you generalize your argument to arbitrarily many?

After we've developed some more technology, statements like this (and more general ones) will become almost automatic.