Math 4803/8803 Homework 7

Due at the beginning of class on **Friday, October 9**.

(1) Let A be an integral domain with fraction field K, and let L/K be a finite extension of degree n. For an A-submodule $M \subset L$, the *dual of M with respect to the trace pairing* is

$$M^* = \{x \in L \mid \mathrm{Tr}_{L/K}(xy) \in A \text{ for all } y \in M\}.$$

(a) Prove that M^* is also an A-submodule of L.
(b) If $M \subset M' \subset L$ are two A-submodules, show that $M'^* \subset M^*$.
(c) Let A' be the integral closure of A in L. Prove that $A' \subset A'^*$ provided that A is integrally closed.

From now on we assume that M is a free A-module of rank n.
(d) Prove that an A-basis for M is a K-basis for L.
(e) Let $e_1, \ldots, e_n$ be a basis for M, and let $x_1, \ldots, x_n \in L$ be the dual basis with respect to the trace pairing. Show that M^* is a free A-module with basis $x_1, \ldots, x_n$.
(f) Show that $M^{**} = M$.

This exercise is meant to motivate the proof of Theorem 2.7.1 in Samuel, which we covered in class on 9/28. (Take $M = A[x] \subset A'$, so $A' \subset A'^* \subset M^*$.) This construction will appear again when we define the different ideal.

(2) Let $f(X) = X^5 - X + 1 \in \mathbf{Q}[X]$, let $x \in \mathbf{C}$ be a root of f, and let $K = \mathbf{Q}(x)$.
(a) Prove that f is irreducible. [Reduce mod 3 and use brute force, or reduce mod 5 and use a clever Galois theory argument, noting that $f(X+1) \equiv f(X) \mod 5$.]
(b) Calculate D_K, the discriminant of K.
(c) Prove that $\mathscr{O}_K = \mathbf{Z}[x]$.

(3) Let $n \geq 2$ and $a \neq 0$ be integers, let $f(X) = X^n - a$, let $\alpha \in \mathbf{C}$ be a root of f, and let $K = \mathbf{Q}(\alpha)$. Note that f may be reducible.
(a) Show that $N_{K/\mathbf{Q}}(f'(\alpha))$ is an integer which is divisible by the discriminant D_K.
(b) Show that $N_{K/\mathbf{Q}}(f'(\alpha))$ divides $n^n a^{n-1}$. [First show the quotient is an algebraic integer.]
(c) Prove that every prime factor of D_K divides na.

(4) Let $n \geq 2$, let p be a prime, and let $K = \mathbf{Q}(\zeta_{p^n})$. We will follow the proof in class or in §2.9 of Samuel to show that $\mathscr{O}_K = \mathbf{Z}[\zeta_{p^n}]$.
(a) Show that $\Phi_{p^n}(X) = \Phi_p(X^{p^{n-1}})$ and that Φ_{p^n} is irreducible over $\mathbf{Q}$ of degree $p^{n-1}(p-1)$.
(b) Compute $\mathrm{Tr}_{K/\mathbf{Q}}(1)$, $\mathrm{Tr}_{K/\mathbf{Q}}(\zeta_{p^n})$, $\mathrm{Tr}_{K/\mathbf{Q}}(1-\zeta_{p^n})$, and $N_{K/\mathbf{Q}}(1-\zeta_{p^n})$.
(c) Prove that $(1-\zeta_{p^n})\mathscr{O}_K \cap \mathbf{Z} = p\mathbf{Z}$, and that $\mathrm{Tr}_{K/\mathbf{Q}}(y(1-\zeta_{p^n})) \in p\mathbf{Z}$ for all $y \in \mathscr{O}_K$.
(d) Prove that $\mathscr{O}_K = \mathbf{Z}[\zeta_{p^n}]$.