Math 4803/8803 Homework 8

Due at the beginning of class on **Wednesday, October 21**.

(1) Let M be a noetherian A-module and let $u\colon M \to M$ be a surjective homomorphism. Show that u is an isomorphism. [Consider the submodules $\ker(u^n)$.] Give a counterexample when M is not noetherian.

(2) Let k be a field, let $A = k[\![X]\!]$, and let $K = k(\!(X)\!) = \mathrm{Frac}(A)$. By problem (2) on homework 1, we know that A is a PID.
  (a) Classify all fractional ideals of A.
  (b) Prove that every fractional ideal is invertible.
  (c) What common group is the group of all fractional ideals isomorphic to?

(3) Let $\mathscr{O} \subset \mathbf{C}$ be the subring of all algebraic integers.
  (a) Prove that every nonzero prime ideal of $\mathscr{O}$ is maximal.
  (b) Prove that $\mathscr{O}$ is not noetherian. [Consider the sequence $(2) \subset (2^{1/2}) \subset (2^{1/4}) \subset \cdots$.]

Hence $\mathscr{O}$ is an example of an integrally closed domain in which every nonzero prime ideal is maximal, but which is not a Dedekind domain.

(4) Let K be a field and let $A = K[X^2, X^3] \subset K[X]$.
  (a) Prove that A is noetherian. [Use $K[X^2] \subset A$.]
  (b) Prove that every nonzero prime ideal of A is maximal.
  (c) Prove that A is not integrally closed.

Hence A is an example of a noetherian domain in which every nonzero prime ideal is maximal, but which is not a Dedekind domain.

(5) Let A be an integral domain with fraction field K. Suppose that every nonzero fractional ideal of A is invertible. Hence the set of nonzero fractional ideals forms a group under multiplication.
  (a) Show that for a nonzero fractional ideal $\mathfrak{a} \subset K$, the inverse of $\mathfrak{a}$ is equal to $\mathfrak{a}^* = \{x \in K \mid x\mathfrak{a} \subset A\}$.
  (b) Prove that A is noetherian. [For an ideal $\mathfrak{a} \subset A$, since $\mathfrak{a}\mathfrak{a}^* = A$ there exist $x_1, \ldots, x_n \in \mathfrak{a}$ and $y_1, \ldots, y_n \in \mathfrak{a}^*$ such that $\sum x_i y_i = 1$. Show that $\mathfrak{a} = (x_1, \ldots, x_n)$.]
  (c) Prove that A is integrally closed. [If $x \in K$ is integral over A, first show that $A[x]$ is a fractional ideal, then that $(x)A[x]^* \subset A[x]^*$.]

It is also true that every nonzero prime ideal of A is maximal, so A is necessarily a Dedekind domain.