Math 4803/8803 Homework 9

Due at the beginning of class on Wednesday, October 28.

1. Prove that a Dedekind domain A is a principal ideal domain if and only if it is a unique factorization domain. [To prove $\Longleftarrow$, you actually only need to assume that any nonzero element a is a product of prime elements.]

2. Let A be a Dedekind domain with fraction field K and let $\mathfrak{a}, \mathfrak{b} \subset K$ be fractional ideals, with duals $\mathfrak{a}^*, \mathfrak{b}^*$. Prove, without doing any computations, that $(\mathfrak{a}\mathfrak{b})^* = \mathfrak{a}^*\mathfrak{b}^*$ and that $\mathfrak{a}^{**} = \mathfrak{a}$.

3. Let K be a number field, let $\mathfrak{p} \subset \mathscr{O}_K$ be a nonzero prime ideal, and let p be the prime number such that $p\mathbf{Z} = \mathfrak{p} \cap \mathbf{Z}$. Prove that the field $k = \mathscr{O}_K/\mathfrak{p}$ is a finite extension field of $\mathbf{F}_p$, and that

$$\mathrm{N}(\mathfrak{p}) = p^{[k\,:\,\mathbf{F}_p]}.$$

4. Let $K = \mathbf{Q}(\sqrt{d})$ be a quadratic number field. Let $\tau \in \mathrm{Gal}(K/\mathbf{Q})$ be the automorphism sending $\sqrt{d}$ to $-\sqrt{d}$, for $x \in K$ set $\overline{x} := \tau(x)$, and for an ideal $\mathfrak{a} \subset \mathscr{O}_K$ set $\overline{\mathfrak{a}} = \tau(\mathfrak{a})$. It is a fact that $\mathfrak{a}\overline{\mathfrak{a}} = (n)$ for a unique positive integer n. Prove that $n = \mathrm{N}(\mathfrak{a})$.

5. Let A be any ring.

a) Let M be an A-module, let $\mathfrak{a}$ be an ideal contained in $\mathrm{Ann}(M)$, let $\overline{A} = A/\mathfrak{a}$, and let $\pi\colon A \to \overline{A}$ be the quotient homomorphism. Prove that M is an $\overline{A}$-module via the rule $\pi(a)m := a\,m$.

b) Let M be an A-module and let $\mathfrak{a} \subset A$ be an ideal. Show that $\mathfrak{a}$ is contained in $\mathrm{Ann}(M/\mathfrak{a}M)$. In particular, if $\mathfrak{a}$ is a maximal ideal then $M/\mathfrak{a}M$ is a vector space over $A/\mathfrak{a}$.

c) Let A be an integral domain, let $\mathfrak{p} \subset A$ be a maximal ideal, and suppose that $\dim_{A/\mathfrak{p}}(\mathfrak{p}/\mathfrak{p}^2) \geq 2$.[1] Prove that A is not Dedekind.

d) Use (c) to give another proof that $K[X^2, X^3]$ is not Dedekind.

e) Now let A be a Dedekind domain, let $\mathfrak{p} \subset A$ be a nonzero prime ideal, and suppose that $[\mathfrak{p}]^2 = [A]$ in $C(A)$. Prove that $\mathfrak{p}$ can be generated by two elements.[2]

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[1]The *Zariski tangent space* at the prime ideal $\mathfrak{p}$ is defined to be the vector space $\mathfrak{p}/\mathfrak{p}^2$. In geometric language, the condition $\dim_{A/\mathfrak{p}}(\mathfrak{p}/\mathfrak{p}^2) \geq 2$ says that A is not nonsingular of dimension one at $\mathfrak{p}$.

[2]In fact, any ideal of a Dedekind domain can be generated by two elements, but this requires more commutative algebra background to prove. See Exercise 9.7 in Atiyah–MacDonald's *Commutative Algebra*.