MATH 4803/8803 HOMEWORK 11
SOLUTION NOTE

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The purpose of this note is to prove the estimate in Theorem 1. I'm not sure to whom to attribute this fact. Spencer Tolbert showed me how to do part of the proof, and the rest came from Matt Baker's algebraic number theory course notes. The proof uses nothing beyond single-variable calculus, yet it is quite tricky without being particularly messy.

Theorem 1. *For all* $x, y \in \mathbf{R}$*, if* $|x| \geq 1$ *then*

$$\big[\sin(2y) - (x + x^{-1})\sin(y)\big]^2 \leq x^2 + 6.$$

Remark 2. This is an extremely good estimate. For instance, taking $x = 100$ and $y = -1.59078$, we have

$$x^2 + 6 - \big[\sin(2y) - (x + x^{-1})\sin(y)\big]^2 \sim 0.0015.$$

In fact, this difference can be made *arbitrarily small* as $x \to \infty$; see Remark 3 below. In particular, $x^2 + 6$ is asymptotic to the maximum value of $[\sin(2y) - (x + x^{-1})\sin(y)]^2$ in the sense that the difference is actually $o_x(1)$.

We will use the following notation:

$$\begin{aligned} \alpha = \alpha(x) &:= x + x^{-1} \\ f(x,y) &:= \sin(2y) - \alpha\sin(y) \\ &= \sin(y)\big(2\cos(y) - \alpha\big). \end{aligned}$$

The last equality comes from the double angle formula $\sin(2y) = 2\sin(y)\cos(y)$. We need to show $f(x,y)^2 \leq x^2 + 6$. We have

$$f(x,y)^2 = \sin^2(y)\big(2\cos(y) - \alpha\big)^2 = \big(1 - \cos^2(y)\big)\big(2\cos(y) - \alpha\big)^2. \tag{2.1}$$

Since $\cos(y + \pi) = -\cos(y)$ and $\alpha(-x) = -\alpha(x)$, we have $f(-x, y+\pi)^2 = f(x,y)^2$; thus we may assume $x \geq 1$. Note that $x + x^{-1}$ is increasing for $x \geq 1$ (its derivative is $1 - x^{-2} \geq 0$), so since $1 + 1^{-1} = 2$, we have $\alpha \geq 2$.

For fixed x, the function $y \mapsto f(x,y)$ is smooth and periodic, so it is bounded; hence it achieves a global maximum at a critical point. We calculate

$$\begin{aligned} \frac{\partial}{\partial y} f(x,y) &= 2\cos(2y) - \alpha\cos(y) \\ &= 4\cos^2(y) - \alpha\cos(y) - 2, \end{aligned}$$

using the double angle formula $\cos(2y) = 2\cos^2(y) - 1$. Let y_0 be a critical point of $f(x, \cdot)$ and let $\beta := \cos(y_0)$, so

$$4\beta^2 - \alpha\beta - 2 = 0. \tag{2.2}$$

Writing $p(z) = 4z^2 - \alpha z - 2$, we have $p(z) > 0$ when $|z|$ is large, and $p(0) = -2$, so p has two roots $z_1 < z_2$. Since

$$p(1) = 4 - \alpha - 2 \le 0 \quad \text{and} \quad p\left(-\frac{1}{2x}\right) = \frac{1}{x^2} + \frac{x + x^{-1}}{2x} - 2 = \frac{3}{2}(x^{-2} - 1) \le 0,$$

we have $z_1 \le -1/2x$ and $z_2 \ge 1$. Now, $\beta = \cos(y_0)$ is a root of p, so $\beta = z_1$ or $\beta = z_2$; but $\beta \in [-1, 1]$, so if $\beta = z_2$ then $\beta = 1$, which implies $f(x, y_0)^2 = 0$ by (2.1), which is impossible because 0 is clearly not a maximum of $f(x, \cdot)^2$. Thus $\beta = z_1 \le -1/2x$, so $x^{-1} \le -2\beta$ and

$$x^{-2} \le 4\beta^2. \tag{2.3}$$

Substituting β for $\cos(y_0)$ into (2.1) and using (2.2) gives

$$\begin{aligned} f(x, y_0)^2 &= (1 - \beta^2)(2\beta - \alpha)^2 \\ &= \alpha^2 + 4\beta^2 - 4\alpha\beta - \alpha^2\beta^2 - 4\beta^4 + 4\alpha\beta^3 \\ &= (\alpha^2 - 4\beta^4 - 4\beta^2 + 4) + (\alpha\beta + 2)(4\beta^2 - \alpha\beta - 2) \\ &= \alpha^2 - 4\beta^4 - 4\beta^2 + 4. \end{aligned} \tag{2.4}$$

Using $\alpha^2 = x^2 + x^{-2} + 2$, we have

$$f(x, y_0^2) = x^2 + 6 + (x^{-2} - 4\beta^2 - 4\beta^4) \le x^2 + 6, \tag{2.5}$$

where the final inequality holds by (2.3). This completes the proof.

Remark 3. Let $a, b, c > 0$ satisfy the Pythagorean equation $a^2 + b^2 = c^2$. Then

$$\begin{aligned} & & a &\le c \\ &\Longrightarrow & 2a &\le c + a \\ &\Longrightarrow & 2ab^2 &\le (c + a)b^2 \\ &\Longrightarrow & 2a(c^2 - a^2) &\le (c + a)b^2 \\ &\Longrightarrow & 2a(c - a)(c + a) &\le (c + a)b^2 \\ &\Longrightarrow & 2a(c - a) &\le b^2 \\ &\Longrightarrow & a(c - a) &\le \frac{b^2}{2}. \end{aligned}$$

Applying this observation to $a = \alpha$, $b = \sqrt{32}$, and $c = \sqrt{\alpha^2 + 32}$ gives

$$|\alpha\beta| = -\alpha\beta = \frac{1}{8}\alpha\left(\sqrt{\alpha^2 + 32} - \alpha\right) = \frac{1}{8}a(c - a) \le \frac{1}{8} \cdot \frac{b^2}{2} = 2.$$

It follows that $\beta \to 0$ as $\alpha \to \infty$, or as $x \to \infty$. Referring to (2.5), we see that the difference between the maximum value of $f(x, y)$ and $x^2 + 6$ tends to zero as $x \to \infty$, which explains Remark 2.