Math 4803/8803 Homework 11

Due at the beginning of class on Wednesday, November 11.

1. Let K be a number field of degree $n = r_1 + 2r_2$. Recall that

$$\rho(K) := \left(\frac{4}{\pi}\right)^{r_2} \frac{n!}{n^n} |D_K|^{1/2}.$$

a) Show that for any integral ideal $\mathfrak{a} \subset \mathcal{O}_K$, if $a = \mathrm{N}(\mathfrak{a})$ then $\mathfrak{a} \mid (a)$.

b) Prove that $C(K)$ is generated by the classes of the nonzero prime ideals $\mathfrak{p} \subset \mathcal{O}_K$ with $\mathrm{N}(\mathfrak{p}) \leq \rho(K)$.

c) Prove that any prime ideal in (b) is a prime factor of (p) for $p \leq \rho(K)$ prime. Conclude that there are finitely many such prime ideals.

2. Let $x \in \mathbf{C}$ be a root of $f(X) = X^3 + 2X + 1$, and let $K = \mathbf{Q}(x)$. By problem 3 of homework 6 we know $\mathcal{O}_K = \mathbf{Z}[x]$ and $D_K = -59$.

a) Compute $\rho(K)$.

b) Find all prime ideals $\mathfrak{p} \subset \mathcal{O}_K$ with $\mathrm{N}(\mathfrak{p}) \leq \rho(K)$. It may help to use the isomorphisms

$$\frac{\mathbf{Z}[x]}{(p)} \xleftarrow{\sim} \frac{\mathbf{Z}[X]}{(p, f)} \xrightarrow{\sim} \frac{\mathbf{F}_p[X]}{(f)}$$

for prime numbers p.

c) Prove that $\#C(K) = 1$. [Look for elements of small norm.]

Thus $\mathbf{Z}[x]$ is a PID.

3. For $K = \mathbf{Q}(\sqrt{3})$ and $K = \mathbf{Q}(\sqrt{5})$, draw the image of $\mathcal{O}_K$ in $\mathbf{R}^2$ under the canonical embedding, and find all four fundamental units in $\mathcal{O}_K$.

[Continued on next page...]

4. Let K be a cubic field with $r_1 = r_2 = 1$. We identify K with a subfield of $\mathbf{R}$ via its real embedding.

a) Prove that $\mathcal{O}_K^\times = \{\pm u^n \mid n \in \mathbf{Z}\}$ for some $u \in \mathcal{O}_K^\times$. We call $\pm u, \pm u^{-1}$ the *primitive units*. Show that there is a unique primitive unit $u > 1$, and that $K = \mathbf{Q}(u)$.

b) Let x be the positive square root of u. Show that the conjugates of u are $x^{-1}e^{\pm iy}$ for some real number y. Prove that

$$D(1, u, u^2) = -4\big[\sin(2y) - (x^3 + x^{-3})\sin(y)\big]^2.$$

It is a fact[1] that, for fixed $x > 1$, one has

$$\big[\sin(2y) - (x^3 + x^{-3})\sin(y)\big]^2 \leq x^6 + 6$$

for all $y \in \mathbf{R}$.

c) Prove that $|D_K| \leq 4u^3 + 24$. [Careful: $D_K \neq D(1, u, u^2)$ in general.]

5. For K in problem 2, compute $\mathcal{O}_K^\times$. [Hint: $X^3 + 2X + 1$ has a root $x \sim -0.45340$.]

---

[1] This is an exercise in single-variable calculus (finding the global maximum of a smooth periodic function), but I will be extremely impressed if you are able to prove it. Give it a try and see how good you really are at Calc I!