Math 4803/8803 Homework 12

Due at the beginning of class on Wednesday, November 18.

1. Let K be a number field and let $A = \mathscr{O}_K$. Prove that there exists $f \in A \setminus \{0\}$ such that $\#C(A_f) = 1$.

2. Let K be a number field and let S be a finite set of maximal ideals of $\mathscr{O}_K$. The *ring of S-integers in K* is

$$\mathscr{O}_{K,S} := \{x \in K \mid v_{\mathfrak{p}}(x) \geq 0 \text{ for all } \mathfrak{p} \notin S\}.$$

This is a subring of K containing $\mathscr{O}_K$. The *group of S-units* is $\mathscr{O}_{K,S}^\times$.

a) Let $f \in \mathscr{O}_K \setminus \{0\}$, and let $(f) = \mathfrak{p}_1^{e_1}, \ldots, \mathfrak{p}_r^{e_r}$ be its prime factorization. Let $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$. Prove that $(\mathscr{O}_K)_f = \mathscr{O}_{K,S}$.

b) Prove that $\mathscr{O}_{K,S}^\times$ is a finitely generated abelian group of rank at most $r_1+r_2-1+\#S$, where as usual r_1 and r_2 denote the number of real and pairs of complex embeddings of K. [Hint: consider adding $v_{\mathfrak{p}}\colon K^\times \to \mathbf{Z}$ to the logarithm.]

In fact, a generalization of Dirichlet's unit theorem states that $\mathscr{O}_{K,S}^\times$ has rank exactly $r_1 + r_2 - 1 + \#S$.

3. Let A be an integral domain, let $\mathfrak{a} \subset A$ be an ideal, let $S \subset A \setminus \{0\}$ be a multiplicatively closed subset, and let $A' = S^{-1}A$. Prove that:

a) $\mathfrak{a}A' = \left\{ \dfrac{a}{s} \mid a \in \mathfrak{a}, s \in S \right\}$.

b) If $\mathfrak{p} \subset A$ is prime with $\mathfrak{p} \cap S = \emptyset$, $a \in A$ and $s \in S$, then $a/s \in \mathfrak{p}A'$ if and only if $a \in \mathfrak{p}$. [Careful: remember that there are multiple ways of expressing an element of $S^{-1}A$ as a fraction.]

c) The contraction of a nonzero ideal is nonzero.

d) Extension is compatible with products, in that for any two ideals $\mathfrak{a}, \mathfrak{b} \subset A$,

$$(\mathfrak{a}A')(\mathfrak{b}A') = (\mathfrak{a}\mathfrak{b})A'.$$

e) $\mathfrak{a}A' = A'$ if and only if $S \cap \mathfrak{a} \neq \emptyset$.

f) $A = S^{-1}A$ if and only if $S \subset A^\times$.