Math 4803/8803 Homework 13

Due at the beginning of class on Wednesday, December 2.

This is the final homework assignment. It is a bit longer because you have two weeks (including Thanksgiving break) to work on it.

1. Let A be a Dedekind domain, let $S \subset A \setminus \{0\}$ be a multiplicatively closed subset, and let $A' = S^{-1}A$. Prove that for any two ideals $\mathfrak{a}', \mathfrak{b}' \subset A'$, we have

$$(\mathfrak{a}' \cap A) \cdot (\mathfrak{b}' \cap A) = (\mathfrak{a}'\mathfrak{b}') \cap A.$$

In other words, contraction is compatible with products in the case of localizations of Dedekind domains.

2. Let A be a Dedekind domain, let $S \subset A \setminus \{0\}$ be a multiplicatively closed subset, and let $A' = S^{-1}A$.

a) Let $\mathfrak{p} \subset A$ be a nonzero prime ideal such that $\mathfrak{p} \cap S = \emptyset$. Generalize [Samuel, Proposition 5.1.5] to prove that for any $n \geq 0$, the natural homomorphism

$$A/\mathfrak{p}^n \longrightarrow A'/(\mathfrak{p}A')^n$$

is an isomorphism.

b) Let $\mathfrak{a} \subset A$ be a nonzero contracted ideal. Prove that the natural homomorphism

$$A/\mathfrak{a} \longrightarrow A'/\mathfrak{a}A'$$

is an isomorphism.

3. Let A be a Dedekind domain and let $\mathfrak{a} \subsetneq A$ be a nonzero proper ideal.

a) Let $S = 1 + \mathfrak{a}$. Verify that S is a multiplicatively closed subset, and show that for a prime ideal $\mathfrak{p} \subset A$, we have $\mathfrak{p} \cap S \neq \emptyset$ if and only if $\mathfrak{p} + \mathfrak{a} = A$.

b) Prove that every ideal of $A/\mathfrak{a}$ is principal. [Localize at $S = 1 + \mathfrak{a}$. Show that $\mathfrak{a}$ is contracted and $S^{-1}A$ is semi-local, then use problem 2(b).]

c) Prove that $\mathfrak{a}$ can be generated by two elements. [Apply (2) to the ideal $\mathfrak{a}/aA$ of A/aA for any nonzero element $a \in \mathfrak{a}$.]

4. **a)** Let A be a semi-local Dedekind domain with perfect fraction field F, let K/F be a finite extension, and let B be the integral closure of A in K. Show B is semi-local.

b) Suppose that there were finitely many prime numbers. Use (a) to prove that $\mathbf{Z}[\sqrt{-5}]$ is a PID, and derive a contradiction. Thus there are infinitely many prime numbers.[1]

5. Let A be a Dedekind domain with perfect fraction field F. Let K/F be a finite extension of degree n, and let B be the integral closure of A in K. Let $\mathfrak{b}, \mathfrak{b}'$ be nonzero

[1] Brian Conrad attributes this ridiculous proof to Larry Washington.

ideals of B, with $\mathfrak{b} \mid \mathfrak{b}'$. Show that $\mathrm{N}_{B/A}(\mathfrak{b}) \mid \mathrm{N}_{B/A}(\mathfrak{b}')$, and that if $\mathrm{N}_{B/A}(\mathfrak{b}) = \mathrm{N}_{B/A}(\mathfrak{b}')$ then $\mathfrak{b} = \mathfrak{b}'$.

6. With the notation in (5), let $\mathfrak{b}$ be a nonzero ideal of B.

a) Prove that
$$\mathrm{N}_{B/A}(\mathfrak{b}) = \big(\mathrm{N}_{K/F}(x) \mid x \in \mathfrak{b}\big).$$
[Localize at a nonzero prime ideal of A.]

b) Prove by example that $\mathrm{N}_{B/A}(\mathfrak{b})$ is not necessarily generated by the norms of a given set of generators for $\mathfrak{b}$. [Take $\mathfrak{b} = B$, and suppose that there are two distinct nonzero prime ideals of B which contract to the same ideal of A.]