Math 4803/8803 Final Examination

Due by 5pm on Friday, December 11.

Collaboration on the final is *not* allowed. You may only refer to your notes, the textbook, previous homework assignments, and other course materials.

1. Let K be a number field. Show that $\mathcal{O}_K$ is a PID if and only if every nonzero ideal $\mathfrak{a} \subset \mathcal{O}_K$ contains an element x such that $|\mathrm{N}_{K/\mathbf{Q}}(x)| = \mathrm{N}(\mathfrak{a})$.

2. Let $x \in \mathbf{C}$ be a root of $X^3 + 10X + 1$ and let $y \in \mathbf{C}$ be a root of $X^3 - 8X + 15$. Let $K = \mathbf{Q}(x)$ and $L = \mathbf{Q}(y)$.

a) Show $\mathrm{D}_K = \mathrm{D}_L$, and compute $\mathcal{O}_K$ and $\mathcal{O}_L$.

a) Show that 17 is inert in K, but that 17 splits completely in L.

a) Use (b) to prove $K \not\cong L$.

Hence cubic fields with the same discriminant need not be isomorphic. (In contrast, quadratic fields are classified by their discriminant.)

3. Prove that $\mathbf{Z}[\sqrt{2}]$ is a PID.

4. Let A be a Dedekind domain with fraction field F, let K/F be a finite separable extension, and let B be the integral closure of A in K. Prove that

$$\mathscr{D}_{B/A} = \big(D(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in B\big).$$

[Localize at a prime ideal of A.]

5. Let $K = \mathbf{Q}(\sqrt{-6})$. Determine which prime numbers p split, ramify, and remain inert in $\mathcal{O}_K$. Your answer should only involve the congruence class of p modulo 24 for $p > 3$. [Hint: use quadratic reciprocity.]