Math 6421 Homework 1
Due at the beginning of class on Friday, August 28.

You are encouraged to collaborate on homework assignments! Just remember to write up your proofs separately and to acknowledge your collaborators on your work.

In all exercises, the ground field, if unspecified, may be assumed algebraically closed.

(1) Let $X = \{(t^3, t^4, t^5) \ : \ t \in \mathbf{C}\} \subset \mathbf{A}^3_{\mathbf{C}}$. Prove that
$$X = V(x^3 - yz,\, y^2 - xz,\, z^2 - x^2y).$$
(2) Let $\varepsilon \in \mathbf{C}$ be small and nonzero. Prove that
$$X_\varepsilon = V(x^3 - yz,\, y^2 - xz,\, z^2 - x^2y + \varepsilon) \subset \mathbf{A}^3_{\mathbf{C}}$$
is a finite set.

(3) The affine variety
$$X = V\big(1 + x^3 + y^3 + z^3 = (1 + x + y + z)^3\big) \subset \mathbf{A}^3_{\mathbf{C}}$$
is called a *cubic surface*. Find at least six lines lying on X. (A *line* in $\mathbf{A}^n$ is a translate of a one-dimensional linear space.) It turns out that any projective cubic surface contains exactly 27 lines.

(4) Let $X_1, \ldots, X_n \subset \mathbf{A}^n$ be a (finite) collection of affine varieties. Prove that $X_1 \cup \cdots \cup X_n$ is an affine variety.

(5) Let $X \subset \mathbf{A}^n$ be a finite set of points. Prove that there exists a collection of n polynomials $f_1, \ldots, f_n \in K[x_1, \ldots, x_n]$ such that $X = V(f_1, \ldots, f_n)$. (The number n is both the dimension of $\mathbf{A}^n$ and the number of polynomials to find.)