Math 6421 Homework 2

Due at the beginning of class on Friday, September 4.

In the following exercises, the ground field, if unspecified, is a general algebraically closed field K.

(1) Let $J = (x^2 + y^2 - 1,\ x - 1) \subset K[x, y]$. Find a set of generators for the ideal $I(V(J)) = \sqrt{J}$. Is $\sqrt{J} = J$?

(2) Let X be an affine variety.
  (a) Show that the coordinate ring $A(X)$ is a field if and only if X consists of a single point.
  (b) Show that if X consists of two points, then $A(X) \cong K \times K$ as rings.

(3) Let $X \subset \mathbf{A}^n$ be an arbitrary subset. Prove that $V(I(X)) = \overline{X}$, the closure of X in the Zariski topology.

(4) Find the irreducible components of the following affine varieties:
  (a) $X = V(x - yz,\ xz - y^2) \subset \mathbf{A}^3$.
  (b) $Y = V(x^2 + y^2 + z^2,\ x^2 - y^2 - z^2 + 1) \subset \mathbf{A}^3$.
  Here you may assume K does not have characteristic 2.

(5) Let $X \subset \mathbf{A}^n$ and $Y \subset \mathbf{A}^m$ be irreducible affine varieties. Prove that $X \times Y$ is irreducible.

(6) Let $M_{2\times 3}(K)$ be the space of 2×3 matrices with entries in K. We identify $M_{2\times 3}(K)$ with $\mathbf{A}^6_K$ in the obvious way. Let $X \subset M_{2\times 3}(K)$ be the set of all matrices with rank at most 1. Prove that X is an affine variety. Is it irreducible? What is its dimension?