Math 6421 Homework 3

Due at the beginning of class on Friday, September 11.

(1) Let A be a ring, let $Y \subset \operatorname{Spec}(A)$ be a closed subset, and let $\mathfrak{p} = I(Y)$.
  (a) Prove that Y is irreducible if and only if $\mathfrak{p}$ is prime. In this case, show that $\mathfrak{p}$ is the unique minimal element of Y (with respect to inclusion). The ideal $\mathfrak{p}$, considered as an element of Y, is called the *generic point* of Y.
  (b) With Y and $\mathfrak{p}$ as above, prove that Y is the closure of $\{\mathfrak{p}\}$ in $\operatorname{Spec}(A)$.
  (c) If A is noetherian, prove that Y is an irreducible component of $\operatorname{Spec}(A)$ if and only if $\mathfrak{p}$ is a *minimal* prime ideal of A (with respect to inclusion).

(2) Let $A = \mathbf{Z}[X]$.
  (a) Find all maximal ideals of A, and draw a picture of $\operatorname{Max}(A)$. [Hint: first show that every maximal ideal contains a nonzero integer.]
  (b) Find all prime ideals of A. You may use the fact that any prime ideal not containing a nonzero integer is principal. Can you draw a picture of $\operatorname{Spec}(A)$?
  (c) What is the dimension of $\operatorname{Spec}(A)$? Is it irreducible?

(3) Let $A = \mathbf{C}[\![T]\!][U]$, let $\mathfrak{m} = (TU - 1)$, and let $X = \operatorname{Spec}(A)$ and $Y = V(\mathfrak{m})$.
  (a) Show that $A/\mathfrak{m}$ is isomorphic to the fraction field $\mathbf{C}(\!(T)\!)$ of $\mathbf{C}[\![T]\!]$. Conclude that $\dim(Y) = 0$.
  (b) Show that $\dim(X) \geq 2$.

By Krull's theorem, $\operatorname{codim}_X(Y) = 1$, so in this case, $\dim(Y) + \operatorname{codim}_X(Y) \neq \dim(X)$.

(4) Let A be a ring and $X = \operatorname{Spec}(A)$. For $f \in A$ write $D(f) = X \setminus V(f)$. This is a Zariski-open subset of X, called a *basic open subset*. Prove the following:
  (a) Every Zariski-open subset of X contains $D(f)$ for some $f \in A$.
  (b) $D(f) \cap D(g) = D(fg)$.
  (c) $D(f) = D(g)$ if and only if $\sqrt{(f)} = \sqrt{(g)}$.
  (d) $D(f) = X$ if and only if $f \in A^\times$, and $D(f) = \emptyset$ if and only if f is nilpotent.
  (e) $\bigcup_{i \in I} D(f_i) = X$ if and only if the f_i generate the unit ideal.
  (f) X is quasi-compact, i.e., every open cover has a finite subcover. [Hint: reduce to the case of a cover of the form $X = \bigcup_{i \in I} D(f_i)$. Then show that a finite subset of the f_i generate the unit ideal.]

Quasi-compactness of X is a very important property; it is somewhat surprising that it true even when A is not noetherian!

(5) **(Bonus)** Prove that the function φ of Example 3.5 cannot be written as a quotient of two polynomials on all of U. [In fact, there does not exist non-constant $f \in A(X)$ such that $V(f) \subset V(x_2, x_4)$: consider only the linear terms of the polynomials in the ideal $(f,\, x_1x_4 - x_2x_3)$.]