Math 6421 Homework 4

Due at the beginning of class on Friday, September 18.

In the following exercises, the ground field, if unspecified, is a general algebraically closed field K.

(1) Let X be an irreducible affine variety with function field $K(X) = \mathrm{Frac}(A(X))$.
  (a) Let U_1, U_2 be a nonempty open subsets of X and let $f_1, f_2, g_1, g_2 \in A(X)$ with g_i nonzero on U_i. Show that if $f_1(x)/g_1(x) = f_2(x)/g_2(x)$ for all $x \in U_1 \cap U_2$, then $f_1/g_1 = f_2/g_2$ as elements in $K(X)$.
  (b) Use (a) to define a canonical inclusion $\mathscr{O}(U) \hookrightarrow A(X)$ for any nonempty open set $U \subset X$, and prove that these inclusions are compatible with restriction in the obvious way.
  (c) If $U = \bigcup_{i\in I} U_i$ is an open cover, show that $\mathscr{O}_X(U) = \bigcap_{i\in I} \mathscr{O}_X(U_i)$ in $K(X)$.

(2) Let X be an irreducible affine variety with function field $K(X)$ and let $\varphi \in K(X)$.
  (a) Show that there is a *maximal* (with respect to inclusion) open subset $U \subset X$ such that $\varphi \in \mathscr{O}_X(U)$, identifying $\mathscr{O}_X(U)$ with a subring of $K(X)$ as in (1). We call U the *domain of definition* of φ.
  (b) Let $\varphi = g/f \in K(\mathbf{A}^n)$. Find the domain of definition of φ.
  (c) Let φ be the rational function of Example 3.5 in Gathmann. Find the domain of definition of φ.
  (d) Suppose $X = V(x^2 + y^2 = 1)$, and let $\varphi = \frac{1-y}{x}$. Find the domain of definition of φ.

(3) Let X be a Hausdorff topological space, let $x \in X$, and let A be a ring. Define a presheaf $\mathscr{F}$ on X as follows: $\mathscr{F}(U) = A$ if $x \in U$, and $\mathscr{F}(U) = 0$ otherwise, with the obvious transition maps. Prove that $\mathscr{F}$ is a sheaf, and calculate all stalks of $\mathscr{F}$. We call $\mathscr{F}$ a *skyscraper sheaf*.

(4) Let X be an affine variety and let $Y \subset X$ be a closed subset. For an open set $U \subset X$ let

$$\mathscr{I}_Y(U) = \{ f \in \mathscr{O}_X(U) \ : \ f(y) = 0 \text{ for all } y \in Y \cap U \}.$$

Prove that $\mathscr{I}_Y(U)$ is an ideal in $\mathscr{O}_X(U)$, and that $U \mapsto \mathscr{I}_Y(U)$ is a sheaf of abelian groups.