Math 6421 Homework 5

Due at the beginning of class on Friday, September 25.

In the following exercises, the ground field, if unspecified, is a general algebraically closed field K.

(1) Let $f\colon X \to Y$ be a continuous map of topological spaces, let $\mathscr{F}$ be a sheaf on X, and for $U \subset Y$ define
$$(f_*\mathscr{F})(U) := \mathscr{F}(f^{-1}(U)).$$
Show that $f_*\mathscr{F}$ is a sheaf on Y. It is called the *pushforward sheaf*.

(2) Let X be an affine variety in $\mathbf{A}^n$ and let $x \in X$. Give a natural isomorphism of K-algebras
$$\mathscr{O}_{X,x} \xrightarrow{\sim} \mathscr{O}_{\mathbf{A}^n,x}\big/ I(X)\mathscr{O}_{\mathbf{A}^n,x}.$$
(Here $I(X)\mathscr{O}_{\mathbf{A}^n,x}$ is the ideal in $\mathscr{O}_{\mathbf{A}^n,x}$ generated by the image of $I(X)$.)

(3) Let X be an affine variety, let $A = A(X)$, and let $\mathfrak{X} = \operatorname{Spec}(A)$. Identifying X with $\operatorname{Max}(A)$, we have $X \subset \mathfrak{X}$.
  (a) Prove that $\mathfrak{Y} \mapsto \mathfrak{Y} \cap X$ is a bijection from the set of closed subsets of $\mathfrak{X}$ to the set of closed subsets of X. [Use the Nullstellensatz.] Prove the same statement for open sets.
  (b) Let $\mathfrak{U} \subset \mathfrak{X}$ be an open set and let $U = \mathfrak{U} \cap X$. Give a canonical identification $\mathscr{O}_{\mathfrak{X}}(\mathfrak{U}) \cong \mathscr{O}_X(U)$. [What happens for basic open subsets?]

Therefore, the construction of the structure sheaf on an affine scheme generalizes the construction of regular functions on an affine variety.

(4) Let X be a topological space and let $\mathscr{F}, \mathscr{G}$ be sheaves of abelian groups on X. A *sheaf morphism* is a collection of homomorphisms $f_U\colon \mathscr{F}(U) \to \mathscr{G}(U)$ for open subsets $U \subset X$ such that, for $V \subset U$ and $s \in \mathscr{F}(U)$, we have $f_U(s)|_V = f_V(s|_V)$. The *kernel* of f is defined to be the presheaf $\ker(f)\colon U \mapsto \ker(f_U)$.
  (a) Prove that $\ker(f)$ is a sheaf.
  (b) For $x \in X$, prove that f induces a natural homomorphism on stalks $f_x\colon \mathscr{F}_x \to \mathscr{G}_x$, and show that $\ker(f)_x = \ker(f_x)$.
  (c) Prove that f is injective if and only if f_x is injective for all $x \in X$.