Math 6421 Homework 6
Due at the beginning of class on Friday, October 2.

In the following exercises, the ground field, if unspecified, is a general algebraically closed field K.

(1) Let $X = \operatorname{Spec}(\mathbf{Z}[x])$ and let $U = D(x) \cup D(2)$. Compute $\mathscr{O}_X(U)$.

(2) An *affine conic* in $\mathbf{A}^2$ is the zero locus of a single irreducible polynomial $f \in K[x, y]$ of degree 2.
  (a) Prove that $V(x - y^2) \cong \mathbf{A}^1$ and $V(xy - 1) \cong \mathbf{A}^1 \setminus \{0\}$.
  (b) Prove that $V(xy - 1) \not\cong V(x - y^2)$.
  (c) Assume $\operatorname{char}(K) \neq 2$. Prove that every affine conic $X \subset \mathbf{A}^2$ is isomorphic to $\mathbf{A}^1$ or to $\mathbf{A}^1 \setminus \{0\}$. [First show that there exist either one or two lines L through the origin such that all translates of L meet X in at most one point. Then choose v perpendicular to L, and consider $(L + av) \cap X$ for $a \in \mathbf{A}^1$.]

(3) Let $f\colon X \to Y$ be a morphism of affine varieties and let $f^*\colon A(Y) \to A(X)$ be the corresponding homomorphism of coordinate rings. Are the following statements true or false? Give a proof or counterexample.
  (a) f is surjective implies f^* is injective.
  (b) f^* is injective implies f is surjective.
  (c) f is injective implies f^* is surjective.
  (d) f^* is surjective implies f is injective.

(4) Let X and Y be affine varieties, let $A = A(X)$ and $B = A(Y)$, let $\mathfrak{X} = \operatorname{Spec}(A)$ and $\mathfrak{Y} = \operatorname{Spec}(B)$, and let $g\colon B \to A$ be a K-algebra homomorphism. Let $f\colon X \to Y$ be the associated morphism of varieties, and let $f'\colon \mathfrak{X} \to \mathfrak{Y}$ be the associated morphism of ringed spaces, defined on sets by $f'(\mathfrak{p}) = g^{-1}(\mathfrak{p})$.
  (a) Prove that f' maps $X = \operatorname{Max}(A) \subset \mathfrak{X}$ into $Y = \operatorname{Max}(B) \subset \mathfrak{Y}$.
  (b) Prove that f coincides with the restriction of f' to X.
  (c) Let $\mathfrak{U} \subset \mathfrak{Y}$ be open and let $U = \mathfrak{U} \cap Y$. Prove that $(f')^\sharp\colon \mathscr{O}_{\mathfrak{Y}}(\mathfrak{U}) \to \mathscr{O}_{\mathfrak{X}}(f'^{-1}\mathfrak{U})$ is precomposition with f, under the identifications $\mathscr{O}_{\mathfrak{Y}}(\mathfrak{U}) = \mathscr{O}_Y(U)$ and $\mathscr{O}_{\mathfrak{X}}(f'^{-1}\mathfrak{U}) = \mathscr{O}_X(f^{-1}U)$. (See problem (3) on the previous homework.)