Math 6421 Homework 7

Due at the beginning of class on Friday, October 9.

In the following exercises, the ground field, if unspecified, is a general algebraically closed field K.

Definition. A *scheme* is a locally ringed space X which can be covered by open subsets U such that $(U, \mathscr{O}_X|_U)$ is an affine scheme. A *morphism* of schemes is a morphism as locally ringed spaces.

Definition. Let X be a scheme and let $x \in X$ be a point. The *residue field* of X at x is the field $\kappa(x) := \mathscr{O}_{X,x}/I_x$, where $I_x \subset \mathscr{O}_{X,x}$ is the maximal ideal.

(1) Let X be a scheme and let $Y = \mathrm{Spec}(B)$ be an affine scheme. Prove that $f \mapsto f^\#$ defines a bijection

$$\big\{\text{morphisms } U \to Y\big\} \xrightarrow{\sim} \big\{\text{ring homomorphisms } B \to \mathscr{O}_X(X)\big\}.$$

[Follow the proof of the analogous fact for prevarieties that we covered in class.]

(2) Let X be a scheme and K a field. Show that a morphism $\mathrm{Spec}(K) \to X$ is equivalent to the data of a point $x \in X$ and a homomorphism $\kappa(x) \to K$.

(3) [Gathmann, Exercise 5.8] Show that:
  (a) Every morphism $\mathbf{A}^1 \setminus \{0\} \to \mathbf{P}^1$ extends to a morphism $\mathbf{A}^1 \to \mathbf{P}^1$.
  (b) Not every morphism $\mathbf{A}^2 \setminus \{0\} \to \mathbf{P}^1$ can be extended to a morphism $\mathbf{A}^2 \to \mathbf{P}^1$.

(4) Let X be an affine variety.
  (a) Prove that every morphism $\mathbf{P}^1 \to X$ is constant.
  (b) Let $f, g \in A(X)$, and define $\varphi\colon D(f) \cup D(g) \to \mathbf{P}^1 = \mathbf{A}^1 \cup \{\infty\}$ by

$$\varphi(x) = \begin{cases} \frac{g(x)}{f(x)} & \text{if } f(x) \neq 0 \\ \infty & \text{if } f(x) = 0. \end{cases}$$

  Prove that φ is a morphism.

(5) Let $X = V\big(y^2 = (x-1)(x-2)\cdots(x-(2n+1))\big) \subset \mathbf{A}^2$. Construct a one-point compactification of X similar to the two-point compactification of $V\big(y^2 = (x-1)(x-2)\cdots(x-2n)\big)$ constructed in class or in Gathmann, Example 5.6.