Math 6421 Homework 8

Due at the beginning of class on Friday, October 16.

In the following exercises, the ground field, if unspecified, is a general algebraically closed field K.

(1) Let X be a prevariety and let $Y \subset X$ be a locally closed subprevariety.
  (a) Show that for all $y \in Y$, the map on stalks $\mathscr{O}_{X,y} \to \mathscr{O}_{Y,y}$ is surjective.
  (b) Prove that Y is open if and only if, for all $y \in Y$, the map on stalks $\mathscr{O}_{X,y} \to \mathscr{O}_{Y,y}$ is an isomorphism.

(2) In class we constructed the projective plane $\mathbf{P}^2$ as $X_1 \cup X_2 \cup X_3$, where $X_i = \mathbf{A}^2$ with coordinates x_i, y_i, and where

$$x_2 = \frac{1}{x_1} \quad y_2 = \frac{y_1}{x_1} \quad x_3 = \frac{1}{y_1} \quad y_3 = \frac{x_1}{y_1}.$$

Let $Y_1 = V\big(y_1^2 = (x_1-1)(x_1-2)(x_1-3)\big)$. Construct a closed subvariety $Y \subset \mathbf{P}^2$ such that $Y \cap X_1 = Y_1$ and $Y \setminus X_1$ is one point.

(3) Let X be a topological space.
  (a) Show that X is Hausdorff if and only if the diagonal $\Delta_X = \{(x,x) \mid x \in X\}$ is closed in $X \times X$ in the product topology.
  (b) Let $f\colon X \to Y$ be a continuous map of topological spaces and let $s\colon Y \to X$ be a section, i.e. a continuous map such that $f \circ s = \mathrm{Id}_Y$. Prove that $s(Y) = ((s \circ f) \times \mathrm{Id}_X)^{-1}(\Delta_X)$. Conclude that $s(Y)$ is closed when X is Hausdorff.
  (c) Let $f\colon X \to Y$ be a continuous map of topological spaces and let $\Gamma_f = \{(x, f(x)) \mid x \in X\} \subset X \times Y$, the *graph* of f. Use (b) to prove that Γ_f is closed when X is Hausdorff.

The point of this exercise is to demonstrate that many formal properties of Hausdorff spaces follow directly from condition (a), hence will apply to separated prevarieties.