Math 6421 Homework 9

Due at the beginning of class on Friday, October 23.

In the following exercises, the ground field, if unspecified, is a general algebraically closed field K.

1. Let X be a prevariety and let $\Delta_X = \{(x,x) \mid x \in X\}$ be the diagonal.

a) Prove that Δ_X is a locally closed subvariety of $X \times X$.

b) Define $\delta_X \colon X \to X \times X$ by $\delta_X(x) = (x,x)$. Prove that δ_X is an isomorphism of X onto Δ_X.

c) Conclude that X is separated if and only if δ_X is a closed immersion.

2. Let X and Y be varieties. Show that $X \times Y$ is separated.

3. Let X be a variety and let $U, V \subset X$ be affine open subsets. Show that $U \cap V$ is affine.

4. Let X be the 2-point compactification of $V\big(y^2 = (x-1)(x-2)(x-3)(x-4)\big) \subset \mathbf{A}^2$ constructed in Gathmann, Example 5.6. Prove that X is separated.

5. Let X be a scheme, let $Y \subset X$ be a closed subset, and define a sheaf of ideals $\mathscr{I}$ as in class by

$$\mathscr{I}(U) = \big\{a \in \mathscr{O}_X(U) \mid a_y \in I_y \text{ for all } y \in Y \cap U\big\}.$$

a) Prove that $\mathscr{I}$ is quasi-coherent: that is, for all affine open subsets $U = \mathrm{Spec}(A) \subset X$ and all $f \in A$, we have $\mathscr{I}(D(f)) = \mathscr{I}(U)_f := \mathscr{I}(U)A_f$ inside $A_f = \mathscr{O}_X(D(f))$.

Hence there is a unique closed immersion $g\colon Y' \hookrightarrow X$ such that $\mathscr{I} = \ker(g^\sharp)$.

b) Show that the set underlying Y' is Y.

Identifying Y with Y', we have thus promoted Y to a closed subscheme of X.

c) Prove that Y is *reduced*: i.e. that the structure sheaf $\mathscr{O}_Y$ has no nilpotent elements.

We call Y the *reduced induced closed subscheme*.