Math 6421 Homework 10

Due at the beginning of class on Friday, October 30.

Definition. Let $X \to S$ be an S-scheme, and let $S' \to S$ be a morphism. We call $X' := X \times_S S'$ the *base change* of X to S'; it is an S'-scheme via the second projection.

This is a psychologically useful bit of alternate terminology for a fiber product. The following lemmas about fiber products (proved in class) will be needed in the problems below.

Lemma 1. *Let $f\colon Y \to X$ be a morphism of S-schemes and let $S' \to S$ be a morphism. Then the square*

$$\begin{array}{ccc} Y \times_S S' & \xrightarrow{(f, \mathrm{Id}_{S'})} & X \times_S S' \\ \downarrow & & \downarrow \\ Y & \xrightarrow[f]{} & X \end{array}$$

is Cartesian.

Lemma 2 (Fiber products are compatible with change of base)**.** *Let X and Y be S-schemes, and let $S' \to S$ be a morphism. Let $X' = X \times_S S'$ and $Y' = Y \times_S S'$ be the base changes. Then the square*

$$\begin{array}{ccc} X' \times_{S'} Y' & \longrightarrow & S' \\ {\scriptstyle (f,g)}\downarrow & & \downarrow \\ X \times_S Y & \longrightarrow & S \end{array}$$

is Cartesian, where $f\colon X' \to X$ and $g\colon Y' \to Y$ are projection onto the first coordinate.

1. **a)** Let $f\colon X \to S$ be a morphism of schemes, let $s \in S$, and define $X_s := \operatorname{Spec}(\kappa(s)) \times_S X$, where $\kappa(s)$ is the residue field of s (cf. homework 7, problem 2). Prove that the projection $X_s \to X$ induces a homeomorphism of the topological space underlying X_s onto $f^{-1}(s)$ in its induced topology. Hence $f^{-1}(s)$ naturally has the structure of a scheme; we call it the *fiber* over s. [Note that $f^{-1}(s)$ need not be closed in X if $\{s\}$ is not closed in S.]

b) Now let X and Y be S-schemes, and let $s \in S$. Prove that $(X \times_S Y)_s = X_s \times_{\operatorname{Spec}\kappa(s)} Y_s$. In other words, the fiber product of schemes is fiberwise a fiber product over a field.

2. Let X and Y be S'-schemes, and let $S' \to S$ be a morphism. Define a natural morphism $f\colon X \times_{S'} Y \to X \times_S Y$, and prove that f is a closed immersion.

3. For a scheme Z we let $|Z|$ denote the topological space underlying Z. Let X and Y be S-schemes.

a) Define a natural map of sets

$$f\colon |X \times_S Y| \longrightarrow |X| \times_{|S|} |Y|,$$

where the right hand side is the fiber product in the category of sets.

b) Prove that f is surjective. [Use homework 7, problem 2. If k_1 and k_2 are field extensions of k, why is $k_1 \otimes_k k_2$ not the zero ring?]

c) Give an example to show that f may fail to be injective.

4. Let $\mathscr{P}$ be a property of morphisms of schemes such that:

(1) a closed immersion has $\mathscr{P}$;
(2) a composition of two morphisms having $\mathscr{P}$ has $\mathscr{P}$; and
(3) $\mathscr{P}$ is stable under base extension, in that if $X \to S$ has $\mathscr{P}$ and $S' \to S$ is a morphism then the base change $X \times_S S' \to S'$ has $\mathscr{P}$.

Prove that:

a) A product of morphisms having $\mathscr{P}$ has $\mathscr{P}$: that is, if $f_i\colon X_i \to Y_i$ for $i = 1, 2$ are morphisms of S-schemes and f_1, f_2 have $\mathscr{P}$, then $(f_1, f_2)\colon X_1 \times_S X_2 \to Y_1 \times_S Y_2$ has $\mathscr{P}$.

b) If $X \to S$ and $Y \to S$ have $\mathscr{P}$ then $X \times_S Y \to S$ has $\mathscr{P}$.

c) If $f\colon Y \to X$ and $g\colon X \to S$ are two morphisms, and if $g \circ f$ has $\mathscr{P}$ and g is separated, then f has $\mathscr{P}$. [First show that the graph morphism $\Gamma_f = \mathrm{Id}_Y \times f\colon Y \to Y \times_S X$ is obtained by base extension from the diagonal morphism $\Delta\colon X \to X \times_S X$. Also see homework 8, problem 3.]

5. Let X and Y be separated S-schemes and let $f\colon X \to Y$ be a morphism. Show that f is separated.