Math 6421 Homework 11

Due at the beginning of class on Friday, November 6.

In the following exercises, the ground field, if unspecified, is a general algebraically closed field K.

1. Let $f_i : \mathbf{A}^n \to \mathbf{P}^n$ be the embedding

$$f_i(x_0, \dots, \widehat{x}_i, \dots, x_n) = (x_0 : \cdots : 1 : \cdots : x_n).$$

Let $U_i = f_i(\mathbf{A}^n)$. For $n = 1, 2$, verify that the gluing data

$$\mathbf{A}^n \supset f_i^{-1}(U_i \cap U_j) \xrightarrow{\sim} U_i \cap U_j \xleftarrow{\sim} f_j^{-1}(U_i \cap U_j) \subset \mathbf{A}^n$$

is compatible with our previous construction of $\mathbf{P}^1$ and $\mathbf{P}^2$ by gluing affine spaces.

2. (Gathmann, Exercise 6.13) Let $a = (a_0 : \cdots : a_n) \in \mathbf{P}^n$ be a point. Show that $\{a\}$ is a projective variety, and find explicit *homogeneous* generators for the homogeneous ideal $I_p(\{a\}) \subset K[x_0, \dots, x_n]$.

3. Let $I \subset K[x_0, \dots, x_n]$ be a homogeneous ideal and let

$$I_{\geq d} = \bigoplus_{e \geq d} I \cap K[x_0, \dots, x_n]_e,$$

the sub-ideal generated by the homogeneous polynomials in I of degree at least d. Prove that $V_p(I_{\geq d}) = V_p(I)$.

4. (Gathmann, Exercise 6.30) Let $L_1, L_2 \subset \mathbf{P}^3$ be two disjoint lines, i.e. 1-dimensional linear subspaces (quotients of planes in $\mathbf{A}^4$). Let $a \in \mathbf{P}^3 \setminus (L_1 \cup L_2)$. Show that there is a unique line $L \subset \mathbf{P}^3$ through a that intersects both L_1 and L_2.

5. (Gathmann, Exercise 6.31)

a) Prove that a graded ring R is an integral domain if and only if for all *homogeneous* elements $f, g \in R$ with $fg = 0$, we have $f = 0$ or $g = 0$.

b) Show that a projective variety $X \subset \mathbf{P}^n$ is irreducible if and only if its homogeneous coordinate ring $S(X)$ is an integral domain.