Math 6421 Homework 12

Due at the beginning of class on Friday, November 13.

In the following exercises, the ground field, if unspecified, is a general algebraically closed field K.

1. (Gathmann, Exercise 6.35) Sketch the set of real points of the complex affine curve $X = V(x^3 - xy^2 + 1) \subset \mathbf{A}^2_{\mathbf{C}}$ and compute the points at infinity of its projective closure $\overline{X} \subset \mathbf{P}^2_{\mathbf{C}}$.

2. (Gathmann, Exercise 7.3(b)) If $X \subset \mathbf{P}^n$ is a projective variety, prove that its structure sheaf as defined in Definition 7.1 coincides with the closed subprevariety structure of X in $\mathbf{P}^n$ as defined in Construction 5.12(b).

3. (Gathmann, Exercises 6.32 and 7.4) Let $X, Y \subset \mathbf{P}^n$ be nonempty projective varieties. Show:

a) The dimension of $C(X)$ is $\dim(X) + 1$.

b) If $\dim(X) + \dim(Y) \geq n$ then $X \cap Y \neq \emptyset$.

Hence an intersection of projective varieties in $\mathbf{P}^n$ is never empty unless one would expect it to be empty for dimensional reasons — so e.g. the phenomenon of parallel non-intersecting lines in the plane does not occur in projective space.

c) Let $m, n \in \mathbf{Z}_{\geq 1}$. Prove that $\mathbf{P}^m \times \mathbf{P}^n$ is not isomorphic to $\mathbf{P}^{m+n}$.

4. Let $\varphi : \mathbf{P}^m \times \mathbf{P}^n \to \mathbf{P}^N$ be the Segre embedding. Prove that $\varphi(\mathbf{P}^m \times \mathbf{P}^n)$ is not contained in a proper linear subspace of $\mathbf{P}^N$.

5. Let $L \subset \mathbf{P}^2$ be a line, let $a \in \mathbf{P}^2 \setminus L$, and let $\varphi : \mathbf{P}^2 \setminus \{a\} \to L$ be the projection from a to L. Let $X \subset \mathbf{P}^2$ be a *plane curve*, i.e. a one-dimensional closed subvariety.

a) Show that $\varphi(X) = L$ if $a \notin X$.

b) Find a counterexample to (a) if $a \in X$.