Math 6421 Homework 13

Due at the beginning of class on Friday, November 20.

This is the final homework assignment. Good luck on the final paper!

In the following exercises, the ground field, if unspecified, is a general algebraically closed field K.

1. Let $f_0, \dots, f_r \in K[x_0, \dots, x_m, y_0, \dots, y_n]$ be bihomogeneous polynomials. Prove that the map

$$(x, y) \mapsto (f_0(x, y) : \cdots : f_r(x, y)) \colon \mathbf{P}^m \times \mathbf{P}^n \to \mathbf{P}^r$$

is a morphism. [Use the Segre coordinates.]

2. Let $f : \mathbf{P}^n \to \mathbf{P}^N$ be the degree-d Veronese embedding. Find explicit equations cutting out the image of f.

3. (Gathmann, Exercise 7.31) Recall that a conic in $\mathbf{P}^2$ over a field of characteristic not equal to 2 is the zero locus of an irreducible homogeneous polynomial of degree 2 in $K[x, y, z]$.

a) Considering the coefficients of such polynomials, show that the set of all conics in $\mathbf{P}^2$ can be identified with an open subset U of $\mathbf{P}^5$.

b) Let $a \in \mathbf{P}^2$. Show that the subset of U consisting of all conics passing through a is the zero locus of a linear equation in the homogeneous coordinates of $U \subset \mathbf{P}^5$.

c) Given 5 points of $\mathbf{P}^2$, no three of which lie on a line, show that there is a unique conic in $\mathbf{P}^2$ passing through all these points.

4. **a)** Let S be a graded ring. Show that $\mathrm{Proj}(S) = \emptyset$ if and only if every element of S_+ is nilpotent.

b) Let $\varphi \colon S \to T$ be a graded homomorphism of graded rings (preserving degrees). Let $U = \{\mathfrak{p} \in \mathrm{Proj}(T) \mid \mathfrak{p} \not\supset \varphi(S_+)\}$. Show that U is an open subset of $\mathrm{Proj}(T)$, and that φ determines a natural morphism $f : U \to \mathrm{Proj}(S)$.

c) Suppose that $\varphi_d \colon S_d \to T_d$ is an isomorphism for all $d \geq d_0$, where d_0 is a positive integer. Then prove that $U = \mathrm{Proj}(T)$ and that $f \colon \mathrm{Proj}(T) \to \mathrm{Proj}(S)$ is an isomorphism.

5. Let S be the graded ring $K[x_0, \dots, x_n]$, let $d \geq 1$, and let S' be the graded ring defined by $S'_e = S_e$ if $d \mid e$, and $S'_e = 0$ otherwise.

a) Show that $\mathrm{Proj}(S') \cong \mathbf{P}^n_K$.

b) Let $N = \binom{n+d}{d} - 1$ and $T = K[y_0, \dots, y_N]$. Define $\varphi \colon T \to S'$ by sending the y_i to the degree-d monomials in $K[x_0, \dots, x_n]$. Prove that φ is surjective, and gives rise to the degree-d Veronese embedding $f : \mathbf{P}^n_K \to \mathbf{P}^N_K$ in the manner of 4(b) above.