# Announcements

September 7

- Homework 1.3 is due Friday.
- Quiz on Friday: section 1.3.
- My office hours **only today** are moved to **10–11am**, right after class.
  - As always, TAs' office hours are posted on the website.
  - Also there are links to other resources like Math Lab.

# Section 1.4

## The Matrix Equation $Ax = b$

# Matrix $\times$ Vector

Let $A$ be an $m \times n$ matrix ($m$ rows, $n$ columns)

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \qquad \text{with columns } v_1, v_2, \ldots, v_n$$

## Definition

The product of $A$ with a vector $x$ in $\mathbf{R}^n$ is the linear combination

$$Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} := x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

The output is a vector in $\mathbf{R}^m$.

Note that the number of columns of $A$ has to equal the number of rows of $x$.

## Example

$$\begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 1 \begin{pmatrix} 4 \\ 7 \end{pmatrix} + 2 \begin{pmatrix} 5 \\ 8 \end{pmatrix} + 3 \begin{pmatrix} 6 \\ 9 \end{pmatrix} = \begin{pmatrix} 32 \\ 50 \end{pmatrix}.$$

## Question

Let $v_1, v_2, v_3$ be vectors in $\mathbf{R}^3$. How can you write the vector equation

$$2v_1 + 3v_2 - 4v_3 = \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}$$

in terms of matrix multiplication?

Answer: Let $A$ be the matrix with colums $v_1, v_2, v_3$, and let $x$ be the vector with entries $2, 3, -4$. Then

$$Ax = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = 2v_1 + 3v_2 - 4v_3,$$

so the vector equation is equivalent to the matrix equation

$$Ax = \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}.$$

Let $v_1, v_2, \ldots, v_n$, and $b$ be vectors in $\mathbf{R}^m$. Consider the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b.$$

It is equivalent to the **matrix equation**

$$Ax = b$$

where

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \qquad \text{and} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Conversely, if $A$ is any $m \times n$ matrix, then

$$Ax = b \qquad \text{is equivalent to the vector equation} \qquad x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b$$

where $v_1, \ldots, v_n$ are the columns of $A$, and $x_1, \ldots, x_n$ are the entries of $x$.

# Linear Systems, Vector Equations, Matrix Equations, ...

We now have *four* equivalent ways of writing (and thinking about) linear systems:

1. As a system of equations:

$$\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_1 - \phantom{3}x_2 &= 5 \end{aligned}$$

2. As an augmented matrix:

$$\left(\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & -1 & 5 \end{array}\right)$$

3. As a vector equation ($x_1 v_1 + \cdots + x_n v_n = b$):

$$x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

4. As a matrix equation ($Ax = b$):

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

In particular, *all four have the same solution set*. We will go back and forth freely between these over and over again, for the rest of the semester.

### Definition

A **row vector** is a matrix with one row. The product of a row vector of length $n$ and a (column) vector of length $n$ is

$$\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} := a_1x_1 + \cdots + a_nx_n.$$

This is a scalar.

If $A$ is an $m \times n$ matrix with rows $r_1, r_2, \ldots, r_m$, and $x$ is a vector in $\mathbf{R}^n$, then

$$Ax = \begin{pmatrix} \text{---} & r_1 & \text{---} \\ \text{---} & r_2 & \text{---} \\ & \vdots & \\ \text{---} & r_m & \text{---} \end{pmatrix} x = \begin{pmatrix} r_1x \\ r_2x \\ \vdots \\ r_mx \end{pmatrix}$$

This is a vector in $\mathbf{R}^m$ (again).

## Example

$$\begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \left(\begin{smallmatrix} 4 & 5 & 6 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right) \\ \left(\begin{smallmatrix} 7 & 8 & 9 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right) \end{pmatrix} = \begin{pmatrix} 4 \cdot 1 + 5 \cdot 2 + 6 \cdot 3 \\ 7 \cdot 1 + 8 \cdot 2 + 9 \cdot 3 \end{pmatrix} = \begin{pmatrix} 32 \\ 50 \end{pmatrix}.$$

Note this is the same as before:

$$\begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 1 \begin{pmatrix} 4 \\ 7 \end{pmatrix} + 2 \begin{pmatrix} 5 \\ 8 \end{pmatrix} + 3 \begin{pmatrix} 6 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 \\ 1 \cdot 7 + 2 \cdot 8 + 3 \cdot 9 \end{pmatrix} = \begin{pmatrix} 32 \\ 50 \end{pmatrix}.$$

Now you have *two* ways of computing $Ax$. In the second, you calculate $Ax$ one entry at a time.

Try both and decide which is your favorite!

## Spans and Solutions to Equations

Let $A$ be a matrix with columns $v_1, v_2, \ldots, v_n$:

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$$

**Very Important**

$Ax = b$ has a solution

$$\iff \text{there exist } x_1, \ldots, x_n \text{ such that } A\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = b$$

"if and only if"

$$\iff \text{there exist } x_1, \ldots, x_n \text{ such that } x_1 v_1 + \cdots + x_n v_n = b$$

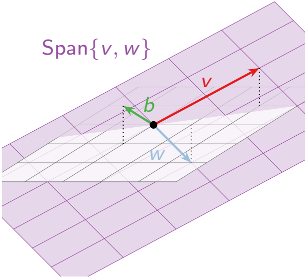
$$\iff b \text{ is a linear combination of } v_1, \ldots, v_n$$

$$\iff b \text{ is in the span of the columns of } A.$$

The last condition is very geometric.

### Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ have a solution?




Columns of $A$:

$$v = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \qquad w = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Solution vector:

$$b = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Is $b$ contained in the span of the columns of $A$?
It sure doesn't look like it.

### Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ have a solution?

Answer: Let's check by solving the matrix equation using row reduction. The first step is to put the system into an augmented matrix.

$$\left(\begin{array}{cc|c} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & -1 & 1 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

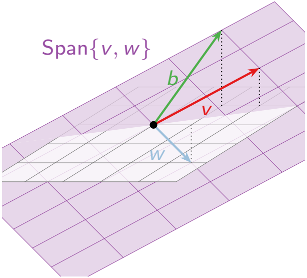
The last equation is $0 = 1$, so the system is *inconsistent*. In other words, the matrix equation

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix} x = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

has no solution, as we guessed.

## Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ have a solution?




Columns of $A$:

$$v = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \qquad w = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Solution vector:

$$b = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

Is $b$ contained in the span of the columns of $A$?
It looks like it. Can you see what $x$ is from the grid on Span$\{v, w\}$?

### Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ have a solution?

Answer: Let's do this systematically using row reduction.

$$\left(\begin{array}{cc|c} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 2 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right)$$

This gives us

$$x = 1 \qquad y = -1.$$

In other words,

$$1\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - 1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \qquad \text{or} \qquad A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

**Poll**

Which of the following true statements can be checked by eyeballing them, *without* row reduction?

A. $\begin{pmatrix}0\\1\\2\end{pmatrix}$ is in the span of $\begin{pmatrix}3\\3\\4\end{pmatrix}, \begin{pmatrix}0\\10\\20\end{pmatrix}, \begin{pmatrix}0\\-1\\-2\end{pmatrix}$.

B. $\begin{pmatrix}0\\1\\2\end{pmatrix}$ is in the span of $\begin{pmatrix}3\\3\\4\end{pmatrix}, \begin{pmatrix}0\\5\\7\end{pmatrix}, \begin{pmatrix}0\\6\\8\end{pmatrix}$.

C. $\begin{pmatrix}0\\1\\2\end{pmatrix}$ is in the span of $\begin{pmatrix}3\\3\\4\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\\sqrt{2}\end{pmatrix}$.

D. $\begin{pmatrix}0\\1\\2\end{pmatrix}$ is in the span of $\begin{pmatrix}5\\7\\0\end{pmatrix}, \begin{pmatrix}6\\8\\0\end{pmatrix}, \begin{pmatrix}3\\3\\4\end{pmatrix}$.

# When Solutions Always Exist

Here are criteria for a linear system to always have a solution.

### Theorem

Let $A$ be an $m \times n$ (non-augmented) matrix. The following are equivalent (either they're all true, or they're all false):

1. $Ax = b$ has a solution for all $b$ in $\mathbf{R}^m$.
2. The span of the columns of $A$ is all of $\mathbf{R}^m$.
3. A has a pivot in each row.

Why is (1) the same as (2)? Why is (1) the same as (3)? If $A$ has a pivot in each row then its reduced row echelon form looks like this:

$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \end{pmatrix} \quad \text{and } (\,A \mid b\,) \text{ reduces to this:} \quad \left(\begin{array}{ccccc|c} 1 & 0 & \star & 0 & \star & | \\ 0 & 1 & \star & 0 & \star & b' \\ 0 & 0 & 0 & 1 & \star & | \end{array}\right).$$

There's no $b$ that makes it inconsistent, so there's always a solution. If $A$ doesn't have a pivot in each row, then its reduced form looks like this:

$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and this can be made inconsistent:} \quad \left(\begin{array}{ccccc|c} 1 & 0 & \star & 0 & \star & 0 \\ 0 & 1 & \star & 0 & \star & 0 \\ 0 & 0 & 0 & 0 & 0 & 16 \end{array}\right).$$

## Properties of the Matrix–Vector Product

Let $c$ be a scalar, $u, v$ be vectors, and $A$ a matrix.

- $A(u + v) = Au + Av$
- $A(cv) = cAv$

See Lay, §1.4, Theorem 5.

For instance, $A(3u - 7v) = 3Au - 7Av$.

Consequence: If $u$ and $v$ are solutions to $Ax = 0$, then so is every vector in Span$\{u, v\}$. Why?

$$\begin{cases} Au = 0 \\ Av = 0 \end{cases} \implies A(xu + yv) = xAu + yAv = x0 + y0 = 0.$$

(Here 0 means the zero vector.)

**Important**

The set of solutions to $Ax = 0$ is a span.