te in w/ lin combo ? span

,$x_n = b$

ds in

ps until Quiz

The professor in your widgets and gizmos class is trying to decide between three different grading schemes for computing your final course grade. The schemes are based on homework (HW), quiz grades (Q), midterms (M), and a final exam (F). The three schemes can be described by the following matrix A:

$$\begin{array}{r c} & \begin{array}{cccc} \text{HW} & \text{Q} & \text{M} & \text{F} \end{array} \\ \begin{array}{r} \text{Scheme 1} \\ \text{Scheme 2} \\ \text{Scheme 3} \end{array} & \begin{pmatrix} 0.1 & 0.1 & 0.5 & 0.3 \\ 0.1 & 0.1 & 0.4 & 0.4 \\ 0.1 & 0.1 & 0.6 & 0.2 \end{pmatrix} = A \\ & \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \end{array}$$

1. Suppose that you have a score of x_1 on homework, x_2 on quizzes, x_3 on midterms, and x_4 on the final, with potential final course grades of b_1, b_2, b_3. Write a matrix equation $Ax = b$ to relate your final grades to your scores.

$$\underbrace{\begin{pmatrix} 0.1 & 0.1 & 0.5 & 0.3 \\ 0.1 & 0.1 & 0.4 & 0.4 \\ 0.1 & 0.1 & 0.6 & 0.2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}}_{\vec{b}}$$

2. Suppose that you end up with averages of 90% on the homework, 90% on quizzes, 85% on midterms, and a 95% score on the final exam. Use Problem 1 to determine which grading scheme leaves you with the highest overall course grade.

$$\begin{pmatrix} 0.1 & 0.1 & 0.5 & 0.3 \\ 0.1 & 0.1 & 0.4 & 0.4 \\ 0.1 & 0.1 & 0.6 & 0.2 \end{pmatrix} \begin{pmatrix} .9 \\ .9 \\ .85 \\ .95 \end{pmatrix} = \begin{pmatrix} \sim .89 \\ .9 \\ \sim .88 \end{pmatrix} \rightarrow \begin{array}{ll} 89\% & \text{scheme 1} \\ 90\% & \text{scheme 2} \\ 88\% & \text{scheme 3} \end{array}$$

3. **a)** Keeping $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ as a general vector, write the augmented matrix $(A \mid b)$.

$$\left(\begin{array}{cccc|c} 0.1 & 0.1 & 0.5 & 0.3 & b_1 \\ 0.1 & 0.1 & 0.4 & 0.4 & b_2 \\ 0.1 & 0.1 & 0.6 & 0.2 & b_3 \end{array}\right)$$

$R_2 = R_1 - R_2 \;/\; R_3 = R_1 - R_3$

b) Row reduce this matrix until you reach row echelon form.

$$\left(\begin{array}{cccc|c} .1 & .1 & .5 & .3 & b_1 \\ 0 & 0 & .1 & -.1 & b_1 - b_2 \\ 0 & 0 & -.1 & .1 & b_1 - b_3 \end{array}\right) \xrightarrow{R_3 = R_2 + R_3} \left(\begin{array}{cccc|c} .1 & .1 & .5 & .3 & b_1 \\ 0 & 0 & .1 & -.1 & b_1 - b_2 \\ 0 & 0 & 0 & 0 & 2b_1 - b_2 - b_3 \end{array}\right)$$

c) Looking at the final matrix in (b), what equation in terms of b_1, b_2, b_3 must be satisfied in order for $Ax = b$ to have a solution?

$2b_1 - b_2 - b_3 = 0$ or else column 5 would be a pivot column. and $0 \neq$ non-zero #

d) The answer to (c) also defines the span of the columns of A. Describe the span geometrically.

The span of the columns of A is all vectors $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ st $(A \mid b)$ is consistent.

$\Downarrow$

$2b_1 - b_2 - b_3 = 0$

[In this case b_1 must be the average of b_2 & b_3]

There are 2 pivots so the corresponding coluns are linearly independant so the span is a plane ($\mathbb{R}^2$)

e) Solve the equation in (c) for b_1. Looking at this equation, is it possible for b_1 to be the largest of b_1, b_2, b_3? In other words, is it ever possible for the grade under Scheme 1 to be the highest of the three final course grades? Why or why not? Which scheme would you argue for?

$$b_1 = \frac{b_2 + b_3}{2}$$

b_1 is the average of b_2 & b_3. At best b_1 can be equal to b_2 & b_3 if $b_2 = b_3$. Otherwise, either b_2 or b_3 will be greater than b_1. Either b_2 or b_3 would be better grading schemes.

Given vectors $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$, $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \in \mathbb{R}^n$ and $A_{m \times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ and scalar $c \in \mathbb{R}$

Matrix Properties

$$A(\vec{v}+\vec{u}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} v_1+u_1 \\ v_2+u_2 \\ \vdots \\ v_n+u_n \end{pmatrix} = \begin{pmatrix} a_{11}(v_1+u_1)+a_{12}(v_2+u_2)+\ldots+a_{1n}(v_n+u_n) \\ a_{21}(v_1+u_1)+a_{22}(v_2+u_2)+\ldots+a_{2n}(v_n+u_n) \\ \vdots \\ a_{m1}(v_1+u_1)+a_{m2}(v_2+u_2)+\ldots+a_{mn}(v_n+u_n) \end{pmatrix}$$

$$= \begin{pmatrix} (a_{11}v_1+a_{12}v_2+\cdots+a_{1n}v_n)+(a_{11}u_1+a_{12}u_2+\ldots+a_{1n}u_n) \\ (a_{21}v_1+a_{22}v_2+\ldots+a_{2n}v_n)+(a_{21}u_1+a_{22}u_2+\ldots+a_{2n}u_n) \\ \vdots \\ (a_{m1}v_1+a_{m2}v_2+\ldots+a_{mn}v_n)+(a_{m1}u_1+a_{m2}u_2+\ldots+a_{mn}u_n) \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = A\vec{v} + A\vec{u} \quad //$$

Notation

(ex) $c \in \mathbb{R}$ can be read as "scalar c in the set of real #'s"

(ex) $\vec{v} \in \mathbb{R}^n$ "a vector of dimension n in n-dimensional space"

$$A(c\vec{v}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_3 \end{pmatrix} = \begin{pmatrix} a_{11}(cv_1)+a_{12}(cv_2)+\ldots+a_{1n}(cv_n) \\ a_{21}(cv_1)+a_{22}(cv_2)+\ldots+a_{2n}(cv_n) \\ \vdots \\ a_{m1}(cv_1)+a_{m2}(cv_2)+\ldots+a_{mn}(cv_n) \end{pmatrix}$$

Constant c in all terms so we can factor it out.

$$= c \begin{pmatrix} a_{11}v_1+a_{12}v_2+\ldots+a_{1n}v_n \\ a_{21}v_1+a_{22}v_2+\ldots+a_{2n}v_n \\ \vdots \\ a_{m1}v_1+a_{m2}v_2+\ldots+a_{mn}v_n \end{pmatrix} = c \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = cA\vec{v}$$

A few more main points (The theory)

$A\vec{x} = \vec{b}$ $\quad A = \begin{pmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{pmatrix}$

$\Updownarrow$

$(A \mid \vec{b})$

$\Updownarrow$

$\vec{a}_1 x_1 + \vec{a}_2 x_2 + \ldots + \vec{a}_n x_n = \vec{b}$

$\Updownarrow$

System of linear equations.

$A\vec{x} = b$ has a solution $\Leftrightarrow$ $\vec{b}$ is in the $\underbrace{\text{column span of } A}_{\text{span of the columns of } A}$

$\Leftrightarrow$ $\vec{b}$ is a linear combination of the columns of A.

$A\vec{x} = \vec{b}$ has a solution $\vec{x}$ for every $\vec{b}$ $\Leftrightarrow$ A has a pivot in every row

$\Downarrow$

A is consistent.