# Announcements

September 14

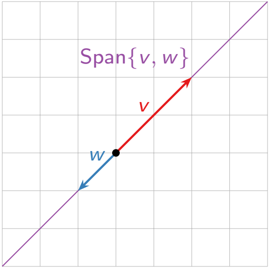
- Homework 1.4 is due Friday.
- Quiz on Friday: section 1.4.
- My office hours are today, 1–2pm and tomorrow, **rescheduled to a3–4pm**, in Skiles 221.
  - As always, TAs' office hours are posted on the website.
  - Also there are links to other resources like Math Lab.
- The first midterm will take place during recitation on Friday, September 30. It covers Chapter 1, sections 1–5 and 7–9.

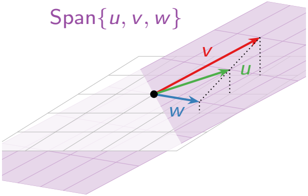
# Section 1.7

## Linear Independence

# Motivation

Sometimes the span of a set of vectors is “smaller” than you expect.




This can mean many things. For example, it can mean you’re using too many vectors to write your solution set.

Notice in each case that one vector in the set is already in the span of the others—so it doesn’t make the span bigger.

Today we will formalize this idea in the concept of *linear (in)dependence*.

# Linear Independence

**Definition**

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ in $\mathbf{R}^n$ is **linearly independent** if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$$

has only the trivial solution $x_1 = x_2 = \cdots = x_p = 0$. The set $\{v_1, v_2, \ldots, v_p\}$ is **linearly dependent** otherwise.

In other words, $\{v_1, v_2, \ldots, v_p\}$ is linearly dependent if there exist numbers $x_1, x_2, \ldots, x_p$, not all equal to zero, such that

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0.$$

This is called a **linear dependence relation**.

Like span, linear (in)dependence is another one of those big vocabulary words that you absolutely need to learn. Much of the rest of the course will be built on these concepts, and you need to know exactly what they mean in order to be able to answer questions on quizzes and exams (and solve real-world problems later on).

Suppose that one of the vectors $\{v_1, v_2, \ldots, v_p\}$ is a linear combination of the other ones (that is, it is in the span of the other ones):

$$v_3 = 2v_1 - \frac{1}{2}v_2 + 6v_4$$

Then the vectors are linearly *de*pendent:

$$2v_1 - \frac{1}{2}v_2 - v_3 + 6v_4 = 0.$$

Conversely, if the vectors are linearly dependent

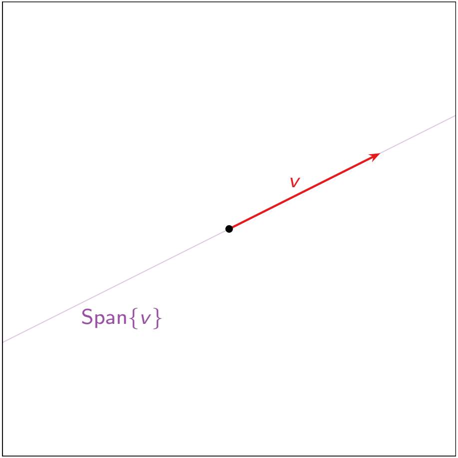
$$2v_1 - \frac{1}{2}v_2 + 6v_4 = 0.$$

then one vector is a linear combination of (in the span of) the other ones:

$$v_2 = 4v_1 + 12v_4.$$

## Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly *de*pendent if and only if one of the vectors is in the span of the other ones.




In this picture

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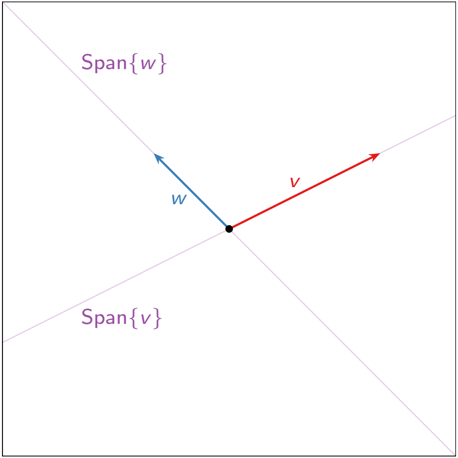
One vector $\{v\}$:
Linearly independent if $v \neq 0$.

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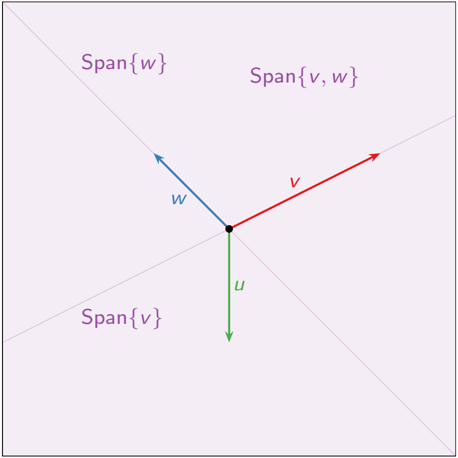
One vector $\{v\}$:
Linearly independent if $v \neq 0$.

Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.

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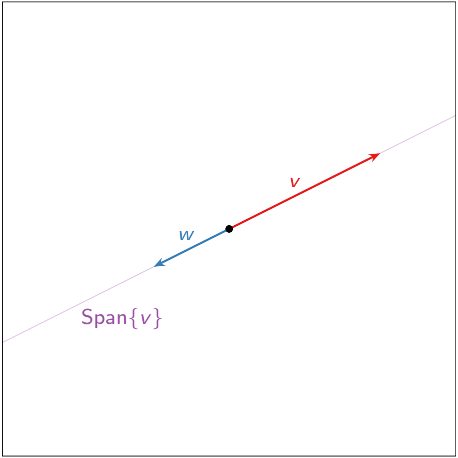
One vector $\{v\}$:
Linearly independent if $v \neq 0$.

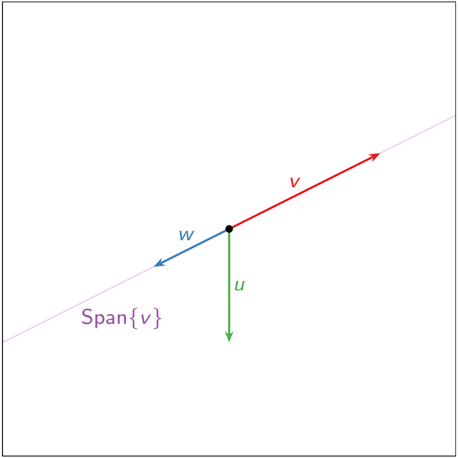
Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.

Three vectors $\{v, w, u\}$:
Linearly dependent: $u$ is in Span$\{v, w\}$.

Also $v$ is in Span$\{u, w\}$ and $w$ is in Span$\{u, v\}$.




Two collinear vectors $\{v, w\}$:
Linearly dependent: $w$ is in Span$\{v\}$ (and vice-versa).

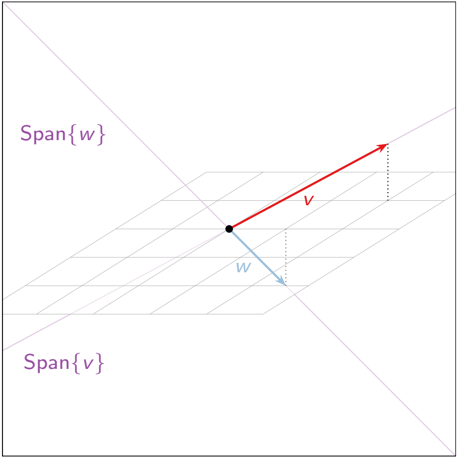



Two collinear vectors $\{v, w\}$:
Linearly dependent: $w$ is in Span$\{v\}$ (and vice-versa).

Observe: *Two* vectors are linearly *dependent* if and only if they are *collinear*.

Three vectors $\{v, w, u\}$:
Linearly dependent: $w$ is in Span$\{v\}$ (and vice-versa).

Observe: If a set of vectors is linearly dependent, then so is any set of vectors that contains it!

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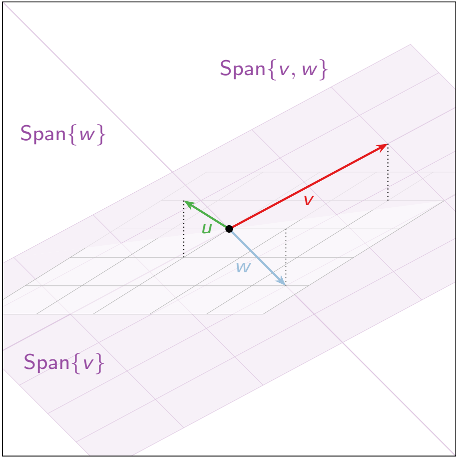
Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.

# Linear Independence

## Pictures in $\mathbf{R}^3$

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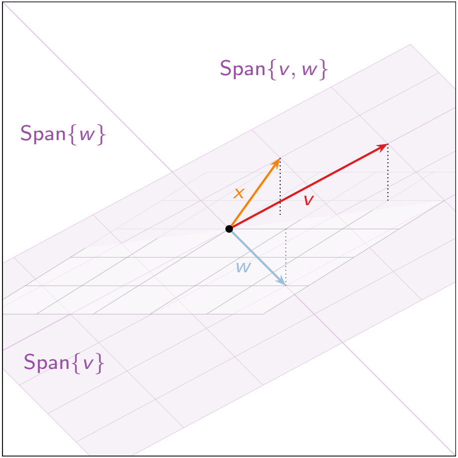
Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.

Three vectors $\{v, w, u\}$:
Linearly independent: no one is in the span of the other two.




In this picture

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Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.

Three vectors $\{v, w, x\}$:
Linearly dependent: $x$ is in $\mathrm{Span}\{v, w\}$.

**Poll**

Are there four vectors $u, v, w, x$ in $\mathbf{R}^3$ which are linearly dependent, but such that $u$ is *not* a linear combination of $v, w, x$? If so, draw a picture; if not, give an argument.

Yes: actually the pictures on the previous slides provide such an example.

Linear dependence of $\{v_1, \ldots, v_p\}$ means *some* $v_i$ is a linear combination of the others, not *any*.

### Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly *de*pendent if and only if one of the vectors is in the span of the other ones.

Take the largest $j$ such that $v_j$ is in the span of the others. Then $v_j$ is in the span of $v_1, v_2, \ldots, v_{j-1}$. Why? If not ($j = 3$):

$$v_3 = 2v_1 - \frac{1}{2}v_2 + 6v_4$$

Rearrange:

$$v_4 = -\frac{1}{6}\left(2v_1 - \frac{1}{2}v_2 - v_3\right)$$

so $v_4$ works as well, but $v_3$ was supposed to be the last one that was in the span of the others.

### Better Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly dependent if and only if there is some $j$ such that $v_j$ is in $\mathrm{Span}\{v_1, v_2, \ldots, v_{j-1}\}$.

**Theorem**

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly dependent if and only if there is some $j$ such that $v_j$ is in $\mathrm{Span}\{v_1, v_2, \ldots, v_{j-1}\}$.

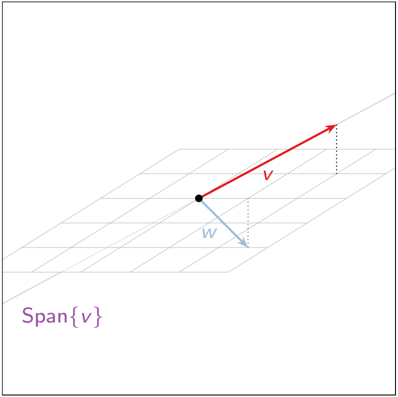
Equivalently, $\{v_1, v_2, \ldots, v_p\}$ is linearly *in*dependent if for every $j$, the vector $v_j$ is not in $\mathrm{Span}\{v_1, v_2, \ldots, v_{j-1}\}$. This means $\mathrm{Span}\{v_1, v_2, \ldots, v_j\}$ is *bigger* than $\mathrm{Span}\{v_1, v_2, \ldots, v_{j-1}\}$.

**Theorem**

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly independent if and only if, for every $j$, the span of $v_1, v_2, \ldots, v_j$ is strictly larger than the span of $v_1, v_2, \ldots, v_{j-1}$.

### Theorem

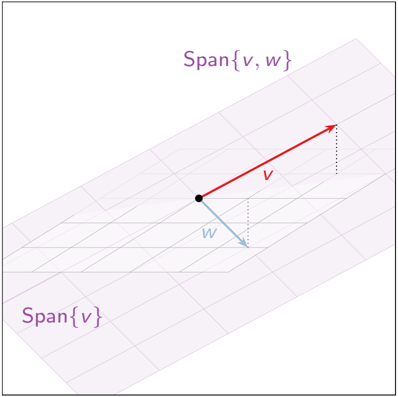
A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly independent if and only if, for every $j$, the span of $v_1, v_2, \ldots, v_j$ is strictly larger than the span of $v_1, v_2, \ldots, v_{j-1}$.




**One vector $\{v\}$:**
Linearly independent: span got bigger (than $(0, 0, 0)$).

### Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly independent if and only if, for every $j$, the span of $v_1, v_2, \ldots, v_j$ is strictly larger than the span of $v_1, v_2, \ldots, v_{j-1}$.

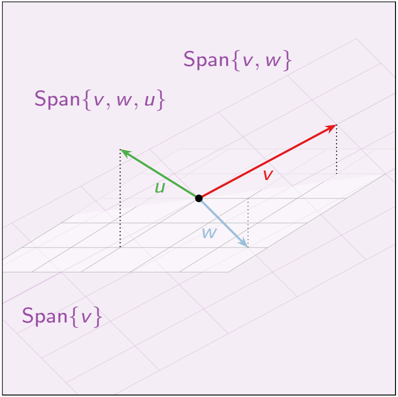



**One vector $\{v\}$:**
Linearly independent: span got bigger (than $(0, 0, 0)$).

**Two vectors $\{v, w\}$:**
Linearly independent: span got bigger.

### Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly independent if and only if, for every $j$, the span of $v_1, v_2, \ldots, v_j$ is strictly larger than the span of $v_1, v_2, \ldots, v_{j-1}$.



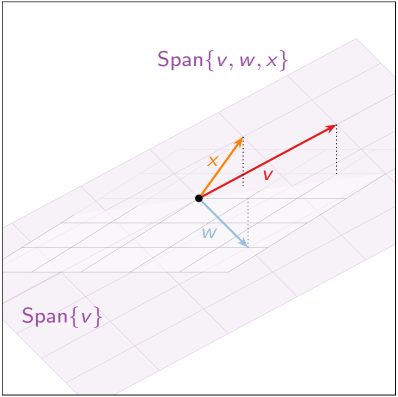

**One vector $\{v\}$:**
Linearly independent: span got bigger (than $(0, 0, 0)$).

**Two vectors $\{v, w\}$:**
Linearly independent: span got bigger.

**Three vectors $\{v, w, u\}$:**
Linearly independent: span got bigger.

## Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly independent if and only if, for every $j$, the span of $v_1, v_2, \ldots, v_j$ is strictly larger than the span of $v_1, v_2, \ldots, v_{j-1}$.




**One vector $\{v\}$:**
Linearly independent: span got bigger (than $(0, 0, 0)$).

**Two vectors $\{v, w\}$:**
Linearly independent: span got bigger.

**Three vectors $\{v, w, x\}$:**
Linearly dependent: span didn't get bigger.

# Checking Linear Independence

Question: Is $\left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\2\end{pmatrix}, \begin{pmatrix}3\\1\\4\end{pmatrix} \right\}$ linearly independent?

Equivalently, does the (homogeneous) the vector equation

$$x\begin{pmatrix}1\\1\\1\end{pmatrix} + y\begin{pmatrix}1\\-1\\2\end{pmatrix} + z\begin{pmatrix}3\\1\\4\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

have a nontrivial solution? How do we solve this kind of vector equation?

$$\begin{pmatrix}1 & 1 & 3\\1 & -1 & 1\\1 & 2 & 4\end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix}1 & 0 & 2\\0 & 1 & 1\\0 & 0 & 0\end{pmatrix}$$

So $x = -2z$ and $y = -z$. So the vectors are linearly *dependent*, and an equation of linear dependence is (taking $z = 1$)

$$-2\begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}1\\-1\\2\end{pmatrix} + \begin{pmatrix}3\\1\\4\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

# Checking Linear Independence

Question: Is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

Equivalently, does the (homogeneous) the vector equation

$$x \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + z \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

have a nontrivial solution?

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 0 & 2 & 4 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The trivial solution $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is the unique solution. So the vectors are linearly *in*dependent.

## Linear Independence and Matrix Columns

In general, $\{v_1, v_2, \dots, v_p\}$ is linearly independent if and only if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$$

has only the trivial solution, if and only if the matrix equation

$$Ax = 0$$

has only the trivial solution, where $A$ is the matrix with columns $v_1, v_2, \dots, v_p$:

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{pmatrix}.$$

This is true if and only if the matrix $A$ has a pivot in each column.

Solving the matrix equation $Ax = 0$ will either verify that the columns $v_1, v_2, \dots, v_p$ of $A$ are linearly independent, or will produce a linear dependence relation.

# Linear Independence

## Two more facts

Fact 1: Say $v_1, v_2, \ldots, v_n$ are in $\mathbf{R}^m$. If $n > m$ then $\{v_1, v_2, \ldots, v_n\}$ is linearly *de*pendent: the matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}.$$

cannot have a pivot in each column (it is too wide).

This says you can't have 4 linearly independent vectors in $\mathbf{R}^3$, for instance.

Fact 2: If one of $v_1, v_2, \ldots, v_n$ is zero, then $\{v_1, v_2, \ldots, v_n\}$ is linearly *de*pendent. For instance, if $v_1 = 0$, then

$$1 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3 + \cdots + 0 \cdot v_n = 0$$

is a linear dependence relation.

# Section 1.8

## Introduction to Linear Transformations

## Motivation

Let $A$ be an $m \times n$ matrix. For the matrix equation $Ax = b$ we have learned to describe

- the solution set: all $x$ in $\mathbf{R}^n$ making the equation true.
- the column span: the set of all $b$ in $\mathbf{R}^m$ making the equation consistent.

It turns out these two sets are very closely related to each other.

In order to understand this relationship, it helps to think of the matrix $A$ as a *transformation* from $\mathbf{R}^n$ to $\mathbf{R}^m$.

It's a special kind of transformation called a *linear transformation*.

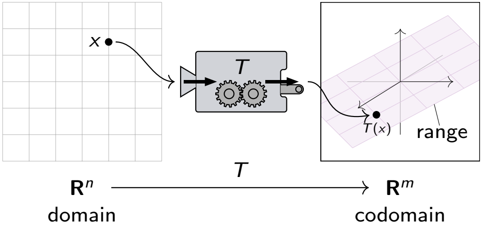
### Definition

A **transformation** (or **function** or **map**) from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a rule $T$ that assigns to each vector $x$ in $\mathbf{R}^n$ a vector $T(x)$ in $\mathbf{R}^m$.

- $\mathbf{R}^n$ is called the **domain** of $T$.
- $\mathbf{R}^m$ is called the **codomain** of $T$.
- For $x$ in $\mathbf{R}^n$, the vector $T(x)$ in $\mathbf{R}^m$ is the **image** of $x$ under $T$.
  Notation: $x \mapsto T(x)$.
- The set of all images $\{T(x) \mid x \text{ in } \mathbf{R}^n\}$ is the **range** of $T$.

Notation:

$$T\colon \mathbf{R}^n \longrightarrow \mathbf{R}^m \quad \text{means} \quad T \text{ is a transformation from } \mathbf{R}^n \text{ to } \mathbf{R}^m.$$




It may help to think of $T$ as a "machine" that takes $x$ as an input, and gives you $T(x)$ as the output.

# Functions from Calculus

Many of the functions you know and love have domain and codomain $\mathbf{R}$.

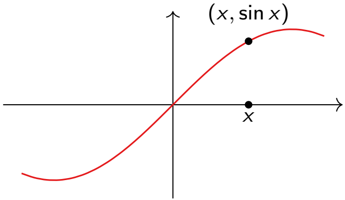
$$\sin\colon \mathbf{R} \longrightarrow \mathbf{R} \qquad \sin(x) = \begin{pmatrix} \text{the length of the opposite edge over the} \\ \text{hypotenuse of a right triangle with angle} \\ x \text{ in radians} \end{pmatrix}$$

Note how I've written down the *rule* that defines the function sin.

$$f\colon \mathbf{R} \longrightarrow \mathbf{R} \qquad f(x) = x^2$$

Note that "$x^2$" is sloppy (but common) notation for a function: it doesn't have a name!

You may be used to thinking of a function in terms of its graph.




The horizontal axis is the domain, and the vertical axis is the codomain.

This is fine when the domain and codomain are $\mathbf{R}$, but it's hard to do when they're $\mathbf{R}^2$ and $\mathbf{R}^3$! You need five dimensions to draw that graph.