Math 1553 Worksheet 4
September 16, 2016

Linear Independence: Concept Questions

1. If three vectors v_1, v_2, v_3 span $\mathbf{R}^3$, must those vectors be linearly independent? Why or why not?

Say $\{v_1, v_2, v_3\}$ is not linearly independant. Then we can say that at least one of the vectors can be expressed as a linear combination of the others. Without loss of triviality, say $v_1 = c_2v_2 + c_3v_3$. v_1 then is in the span of v_2 & v_3 which can at most be a plane ($\mathbb{R}^2$). This is a contradiction with the initial statement. Therefore $\{v_1, v_2, v_3\}$ must be linearly independant

2. Which of the following true statements can be checked without row reduction?

a) $\left\{\begin{pmatrix}3\\3\\4\end{pmatrix}, \begin{pmatrix}0\\0\\\pi\end{pmatrix}, \begin{pmatrix}0\\\sqrt{2}\\0\end{pmatrix}\right\}$ is linearly independent.

can't express the other vectors as a linear combination of the other ones -

b) $\left\{\begin{pmatrix}3\\3\\4\end{pmatrix}, \begin{pmatrix}0\\10\\20\end{pmatrix}, \begin{pmatrix}0\\5\\7\end{pmatrix}\right\}$ is linearly independent.

can't multiply $\vec{v}_2$ by any constant to get $\vec{v}_3$ so both are linearly independant

c) $\left\{\begin{pmatrix}3\\3\\4\end{pmatrix}, \begin{pmatrix}0\\10\\20\end{pmatrix}, \begin{pmatrix}0\\5\\7\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ is linearly dependent.

These vectors are in $\mathbb{R}^3$ so maximum # of vectors that can span $\mathbb{R}^3$ is 3. If you have a fourth vector, at least 1 must be dependant on the others

d) $\left\{\overset{v_1}{\begin{pmatrix}3\\3\\4\end{pmatrix}}, \overset{v_2}{\begin{pmatrix}0\\10\\20\end{pmatrix}}, \overset{v_3}{\begin{pmatrix}0\\0\\0\end{pmatrix}}\right\}$ is linearly dependent.

def'n of lin dep. $c_1v_1 + c_2v_2 = v_3$
- can find c_1 & c_2 (both equal to 0) such that above statement is true

3. How many solutions can the matrix equation $Ax = b$ have if the columns of A are linearly independent? [Try $b = 0$ first.]

a) 0 b) 1 c) ∞.

columns lin. indep.?

✓ $\left(\begin{array}{ccc|c}1&0&0&\\0&1&0&b\\0&0&1&\end{array}\right) \rightarrow$ Unique (1 soln)

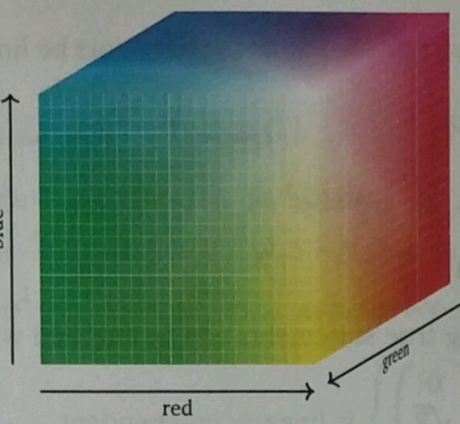
✓ $\left(\begin{array}{ccc|c}1&0&0&\\0&1&0&b\\0&0&1&\\0&0&0&\end{array}\right) \rightarrow$ either unique or no solns

$b = \begin{pmatrix}b_1\\b_2\\b_3\\0\end{pmatrix}$ $\begin{pmatrix}b_1\\b_2\\b_3\\b_4 \neq 0\end{pmatrix}$

X $\left(\begin{array}{ccc|c}1&0&c_1&b\\0&1&c_2&\end{array}\right) \leftarrow$ columns of A not linearly independant

(ex) $c_1\begin{pmatrix}1\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}c_1\\c_2\end{pmatrix}$

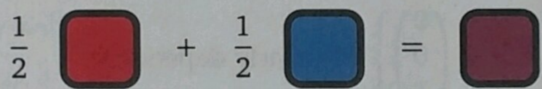
Every color on my computer monitor is a vector in $\mathbf{R}^3$ with coordinates between 0 and 255, inclusive. The coordinates correspond to the amount of red, green, and blue in the color.



Given colors $v_1, v_2, \dots, v_p$, we can form a "weighted average" of these colors by making a linear combination

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$$

with $c_1 + c_2 + \cdots + c_p = 1$. Example:



4. Consider the colors on the right. Are these colors linearly independent? What does this tell you about the colors? If $Ax=0$ results in $x=0$ (trivial) then they are lin. indep.

$$\begin{array}{ccc} v_1 & v_2 & v_3 \\ \begin{pmatrix} 240 \\ 140 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 120 \\ 100 \end{pmatrix} & \begin{pmatrix} 60 \\ 125 \\ 75 \end{pmatrix} \end{array}$$

★ You can mix v_1 & v_2 to get v_3

– What about mixing v_1 & v_3 / w/ v_2 & v_3?

What goes wrong?

$4v_2 - 3v_3 = v_1$

$\frac{4}{3}v_3 - \frac{1}{3}v_1 = v_2$

When you mix colors, you can't remove part of a mixed color

(ex) blue + yellow = green

green − yellow = blue

↑ can't remove yellow, it's mixed

$$\left(\begin{array}{ccc|c} 240 & 0 & 60 & 0 \\ 140 & 120 & 125 & 0 \\ 0 & 100 & 75 & 0 \end{array}\right) \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & \frac{1}{4} \\ 0 & 1 & \frac{3}{4} \\ 0 & 0 & 0 \end{pmatrix}$$

[x_3 free, so all three not linearly independant

x_1 & x_2 are pivots, so lin indep.

5. Consider the colors on the right. For which h is

$$\left\{ \overset{v_1}{\begin{pmatrix} 180 \\ 50 \\ 200 \end{pmatrix}}, \overset{v_2}{\begin{pmatrix} 100 \\ 150 \\ 100 \end{pmatrix}}, \overset{v_3}{\begin{pmatrix} 116 \\ 130 \\ h \end{pmatrix}} \right\}$$

linearly dependent? What does that say about the corresponding color?

$$\begin{pmatrix} 180 \\ 50 \\ 200 \end{pmatrix} \quad \begin{pmatrix} 100 \\ 150 \\ 100 \end{pmatrix}$$

v_1, v_2, v_3 are linearly independant if the solution to $Ax=0$ is $\vec{0}$ and linearly dependant if not.

$h =$	40	80	120	160	200	240
	independant	independant	dependant	independant	independant	independant

↑ You can mix color 1 (v_1) and color 2 (v_2) to get color 3 (v_3) aka can find mixing proportions c_1 & $c_2 \neq 0$ such that $c_1 v_1 + c_2 v_2 = v_3$