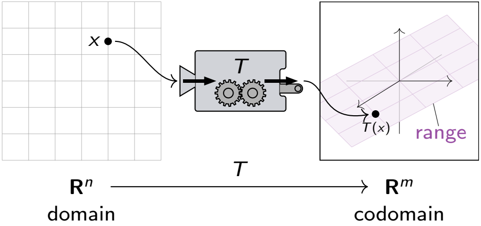
Announcements

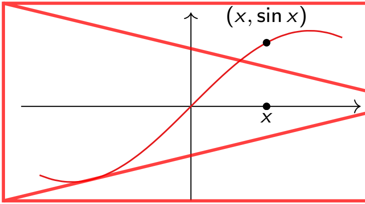
September 19

- Please complete the mid-semester CIOS survey this week.
- The first midterm will take place during recitation a week from Friday, September 30. It covers Chapter 1, sections 1–5 and 7–9.
- Homeworks 1.5, 1.7, 1.8 are due Friday.
  - There are three this week so that there can be two next week, the week of the midterm.
- Quiz on Friday: sections 1.5 and 1.7.
- My office hours are Wednesday, 1–2pm and Thursday, 3:30–4:30pm, in Skiles 221.
  - I'll have extra office hours next week.
  - As always, TAs' office hours are posted on the website.
  - Also there are links to other resources like Math Lab.

Definition

A **transformation** (or **function** or **map**) from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a rule T that assigns to each vector x in $\mathbf{R}^n$ a vector $T(x)$ in $\mathbf{R}^m$.





You may be used to thinking of a function in terms of its graph. But for a function from $\mathbf{R}^m$ to $\mathbf{R}^n$, the graph needs $m + n$ dimensions, so it's hard to visualize and draw. We won't be graphing most of the transformations in this class.

Matrix Transformations

Most of the transformations we encounter in this class will come from a matrix.

Definition

Let A be an $m \times n$ matrix. The **matrix transformation** associated to A is the transformation

$$T\colon \mathbf{R}^n \longrightarrow \mathbf{R}^m \qquad \text{defined by} \qquad T(x) = Ax.$$

In other words, T takes the vector x in $\mathbf{R}^n$ to the vector Ax in $\mathbf{R}^m$.

- The *domain* of T is $\mathbf{R}^n$, which is the number of columns of A.
- The *codomain* of T is $\mathbf{R}^m$, which is the number of rows of A.
- The *range* of T is the set of all images of T:

$$T(x) = Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

This is the column span of A.

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T\colon \mathbf{R}^2 \to \mathbf{R}^3$.

- If $u = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ then $T(u) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 7 \end{pmatrix}$.
- Let $b = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}$. Find v in $\mathbf{R}^2$ such that $T(v) = b$. Is there more than one?

  We want to find v such that $Av = b$. We know how to do that:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} v = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix} \xrightarrow{\text{augmented matrix}} \left(\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 1 & 5 \\ 1 & 1 & 7 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right).$$

  This gives $x = 2$ and $y = 5$, or $v = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ (unique). In other words,

$$T(v) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}.$$

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T\colon \mathbf{R}^2 \to \mathbf{R}^3$.

- Is there any c in $\mathbf{R}^3$ such that there is more than one w $\mathbf{R}^2$ with $T(w) = c$?

  Translation: is there any c in $\mathbf{R}^3$ such that the solution set for $Ax = c$ has more than one vector w in it?

  The solution set to $Ax = b$ has only one vector v. This is a translate of the solution set to $Ax = 0$. So is the solution set to $Ax = c$. So no!

- Find c such that there is *no* v with $T(v) = c$.

  Translation: Find c such that $Ax = c$ is inconsistent.

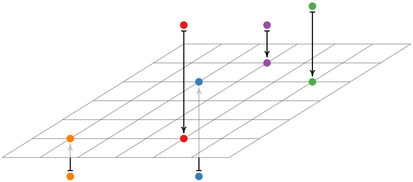
  Translation: Find c not in the column span of A (i.e., the range of T).

  We could draw a picture, or notice: $a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ b \\ a+b \end{pmatrix}$. So anything in the column span has the same first and last coordinate. So $c = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is not in the column span.

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and let $T(x) = Ax$, so $T\colon \mathbf{R}^3 \to \mathbf{R}^3$. Then

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}.$$

This is *projection onto the xy-axis*. Picture:



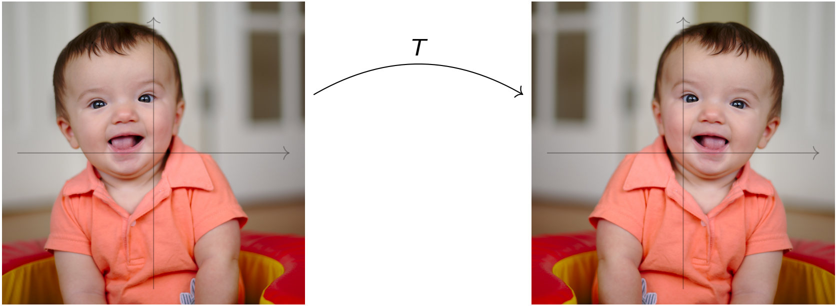
Matrix Transformations

Geometric example

Let $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T\colon \mathbf{R}^2 \to \mathbf{R}^2$. Then

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}.$$

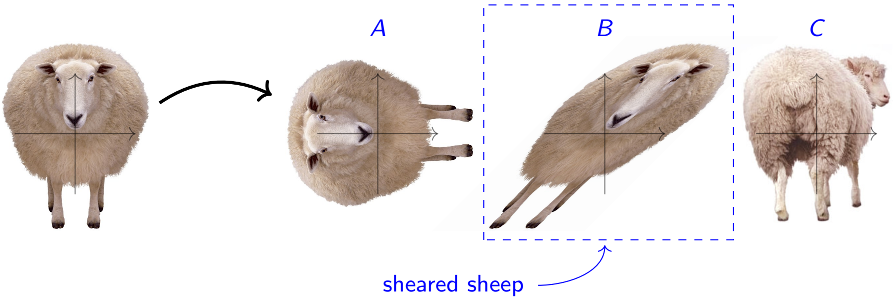
This is *reflection over the y-axis*. Picture:



Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T\colon \mathbf{R}^2 \to \mathbf{R}^2$. ($T$ is called a **shear**.)

Poll

What does T do to this sheep?
Hint: first draw a picture what it does to the box *around* the sheep.



Linear Transformations

Recall: If A is a matrix, u, v are vectors, and c is a scalar, then

$$A(u+v) = Au + Av \qquad A(cv) = cAv.$$

So if $T(x) = Ax$ is a matrix transformation then,

$$T(u+v) = T(u) + T(v) \qquad T(cv) = cT(v).$$

This property is so special that it has its own name.

Definition

A transformation $T : \mathbf{R}^n \to \mathbf{R}^m$ is **linear** if it satisfies the above equations for all vectors u, v in $\mathbf{R}^n$ and all scalars c.

In other words, T "respects" addition and scalar multiplication.

Check: if T is linear, then

$$T(0) = 0 \qquad T(cu + dv) = cT(u) + dT(v)$$

for all vectors u, v and scalars c, d. More generally,

$$T\left(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n\right) = c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n).$$

In engineering this is called **superposition**.

Linear Transformations

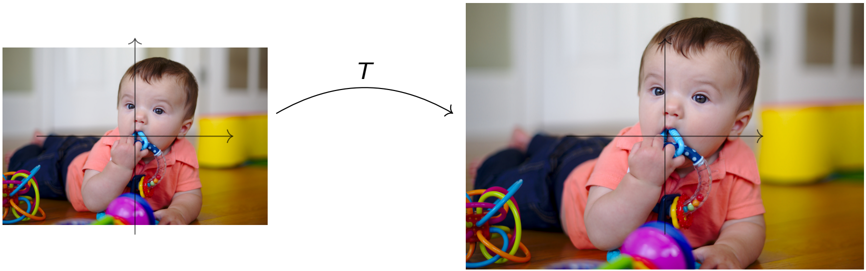
Dilation

Define $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ by $T(x) = 1.5x$. Is T linear? Check:

$$T(u+v) = 1.5(u+v) = 1.5u + 1.5v = T(u) + T(v)$$
$$T(cv) = 1.5(cv) = c(1.5v) = c(Tv).$$

So T satisfies the two equations, hence T is linear.

This is called **dilation** or **scaling** (by a factor of 1.5). Picture:



Define $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ by

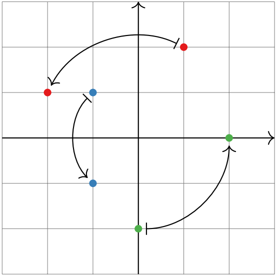
$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}.$$

Is T linear? Check:

$$T\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix} + \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix} = \begin{pmatrix} -(u_2 + v_2) \\ (u_1 + v_1) \end{pmatrix} = T\begin{pmatrix} u_1 + u_2 \\ v_1 + v_2 \end{pmatrix}$$

$$T\left(c\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = T\begin{pmatrix} cv_1 \\ cv_2 \end{pmatrix} = \begin{pmatrix} -cv_2 \\ cv_1 \end{pmatrix} = c\begin{pmatrix} -v_2 \\ v_1 \end{pmatrix} = cT\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

So T satisfies the two equations, hence T is linear. This is called **rotation** (by 90°). Picture:



Section 1.9

The Matrix of a Linear Transformation

Linear Transformations are Matrix Transformations

Definition

The **unit coordinate vectors** in $\mathbf{R}^n$ are

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \ldots, \quad e_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Recall: A matrix A defines a linear transformation T by $T(x) = Ax$.

Theorem

Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. Let

$$A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix}.$$

This is an $m \times n$ matrix, and T is the matrix transformation for A: $T(x) = Ax$.

In particular, *every linear transformation is a matrix transformation.*
The matrix A is called the **standard matrix** for T.

Linear Transformations are Matrix Transformations
Continued

Why? Suppose for simplicity that $T\colon \mathbf{R}^3 \to \mathbf{R}^2$.

$$
\begin{aligned}
T\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= T\left(x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) \\
&= T\big(xe_1 + ye_2 + ze_3\big) \\
&= xT(e_1) + yT(e_2) + zT(e_3) \\
&= \begin{pmatrix} | & | & | \\ T(e_1) & T(e_2) & T(e_3) \\ | & | & | \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
&= A\begin{pmatrix} x \\ y \\ z \end{pmatrix}.
\end{aligned}
$$

So when we think of a matrix as a function from $\mathbf{R}^n$ to $\mathbf{R}^m$, it's the same as thinking of a linear transformation.

Linear Transformations are Matrix Transformations

Example

We defined the **dilation** transformation $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ by $T(x) = 1.5x$. What is its standard matrix?

$$\left.\begin{aligned} T(e_1) &= 1.5e_1 = \begin{pmatrix} 1.5 \\ 0 \end{pmatrix} \\ T(e_2) &= 1.5e_2 = \begin{pmatrix} 0 \\ 1.5 \end{pmatrix} \end{aligned}\right\} \implies A = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix}.$$

Check:

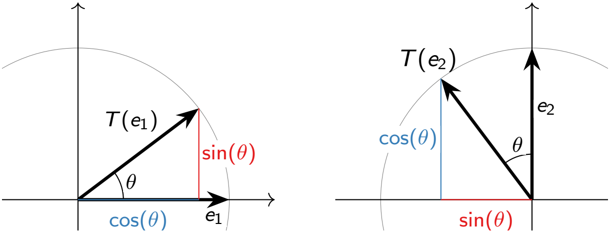
$$\begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1.5x \\ 1.5y \end{pmatrix} = 1.5 \begin{pmatrix} x \\ y \end{pmatrix} = T \begin{pmatrix} x \\ y \end{pmatrix}.$$

Question

What is the matrix for the linear transformation $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$T(x) = x \text{ rotated counterclockwise by an angle } \theta?$$

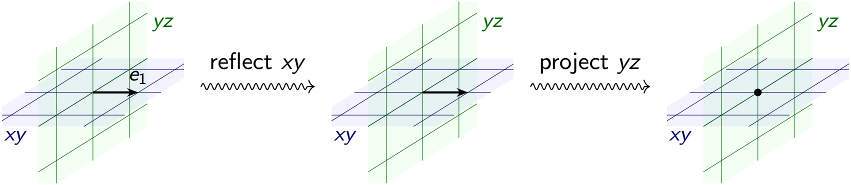
(Check linearity...)



$$\left.\begin{aligned} T(e_1) &= \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \\ T(e_2) &= \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \end{aligned}\right\} \implies A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \qquad \left(\begin{aligned} \theta = 90^\circ &\implies \\ A = &\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{aligned} \right)$$

Question

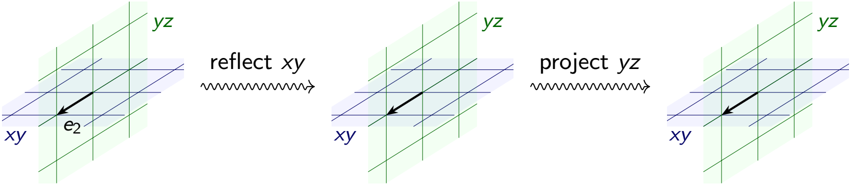
What is the matrix for the linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane?



$$T(e_1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Question

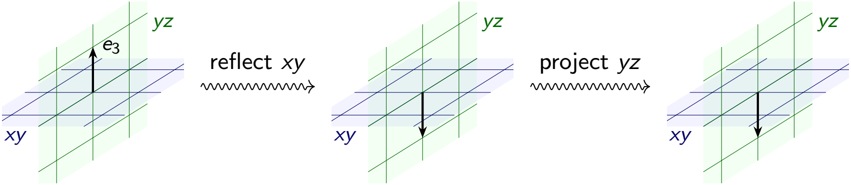
What is the matrix for the linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane?



$$T(e_2) = e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Question

What is the matrix for the linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ that reflects through the *xy*-plane and then projects onto the *yz*-plane?



$$T(e_3) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

Question

What is the matrix for the linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane?

$$\left.\begin{aligned} T(e_1) &= \begin{pmatrix}0\\0\\0\end{pmatrix} \\ T(e_2) &= \begin{pmatrix}0\\1\\0\end{pmatrix} \\ T(e_1) &= \begin{pmatrix}0\\0\\-1\end{pmatrix} \end{aligned}\right\} \implies A = \begin{pmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & -1\end{pmatrix}.$$

Other Geometric Transformations

There is a long list of geometric transformations of $\mathbf{R}^2$ in §1.9 of Lay. (Reflections over the diagonal, contractions and expansions along different axes, shears, projections, . . .) Please look them over.