Announcements

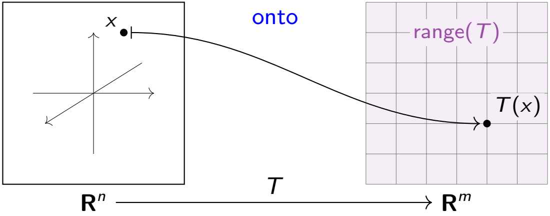
September 21

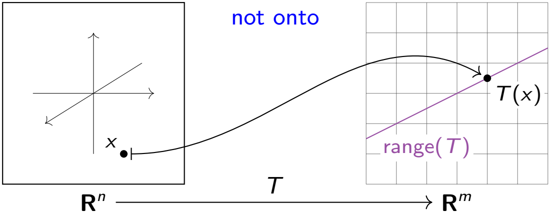
- Please complete the mid-semester CIOS survey this week.
- The first midterm will take place during recitation a week from Friday, September 30. It covers Chapter 1, sections 1–5 and 7–9.
- WeBWorK Assignments 1.5, 1.7, 1.8 are due Friday.
  - There are three this week so that there can be two next week, the week of the midterm.
- Quiz on Friday: sections 1.5 and 1.7.
- My office hours are today, 1–2pm and tomorrow, 3:30–4:30pm, in Skiles 221.
  - I'll have extra office hours next week.
  - As always, TAs' office hours are posted on the website.
  - Also there are links to other resources like Math Lab.

Onto Transformations

Definition

A transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is **onto** (or **surjective**) if the range of T is equal to $\mathbf{R}^m$ (its codomain). In other words, each b in $\mathbf{R}^m$ is the image of *at least one* x in $\mathbf{R}^n$. Note that *not* onto means there is some b in $\mathbf{R}^m$ which is not the image of any x in $\mathbf{R}^n$.





Characterization of Onto Transformations

Theorem

Let $T: \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation with matrix A. Then the following are equivalent:

- T is onto
- $T(x) = b$ has a solution for every b in $\mathbf{R}^m$
- $Ax = b$ is consistent for every b in $\mathbf{R}^m$
- The columns of A span $\mathbf{R}^m$
- A has a pivot in every row.

Question

If $T: \mathbf{R}^n \to \mathbf{R}^m$ is onto, what can we say about the relative sizes of n and m?

Answer: T corresponds to an $m \times n$ matrix A. In order for A to have a pivot in every row, it must have *at least as many* columns as rows: $m \leq n$.

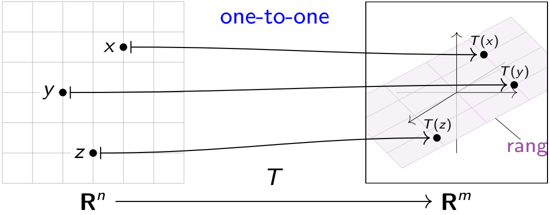
$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \end{pmatrix}$$

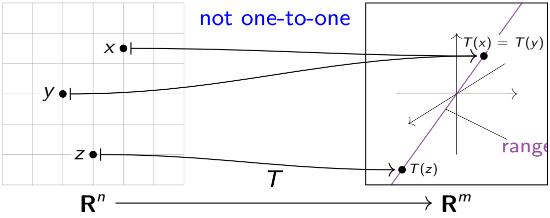
For instance, $\mathbf{R}^2$ is "too small" to map *onto* $\mathbf{R}^3$.

One-to-one Transformations

Definition

A transformation $T : \mathbf{R}^n \to \mathbf{R}^m$ is **one-to-one** (or **into**, or **injective**) if different vectors in $\mathbf{R}^n$ map to different vectors in $\mathbf{R}^m$. In other words, each b in $\mathbf{R}^m$ is the image of *at most one* x in $\mathbf{R}^n$. Note that *not* one-to-one means there are different vectors in $\mathbf{R}^n$ with the same image.





Characterization of One-to-One Transformations

Theorem

Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation with matrix A. Then the following are equivalent:

- T is one-to-one
- $T(x) = b$ has one or zero solutions for every b in $\mathbf{R}^m$
- $Ax = b$ has a unique solution or is inconsistent for every b in $\mathbf{R}^m$
- $Ax = 0$ has a unique solution
- The columns of A are linearly independent
- A has a pivot in every column.

Question

If $T : \mathbf{R}^n \to \mathbf{R}^m$ is one-to-one, what can we say about the relative sizes of n and m?

Answer: T corresponds to an $m \times n$ matrix A. In order for A to have a pivot in every column, it must have *at least as many* rows as columns: $n \leq m$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

For instance, $\mathbf{R}^3$ is "too big" to map *into* $\mathbf{R}^2$.

Chapter 2

Matrix Algebra

Section 2.1

Matrix Operations

Motivation

Recall: we can turn any system of linear equations into a matrix equation

$$Ax = b.$$

This notation is suggestive. Can we solve the equation by "dividing by A"?

$$x \stackrel{??}{=} \frac{b}{A}$$

Answer: sometimes, but you have to know what you're doing.

Today we'll study *matrix algebra*: adding and multiplying matrices.

More Notation for Matrices

Let A be an $m \times n$ matrix.

We write a_{ij} for the entry in the ith row and the jth column. It is called the ***ij*th entry** of the matrix.

$$\begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix}$$

ith row

jth column

The entries $a_{11}, a_{22}, a_{33}, \ldots$ are the **diagonal entries**; they form the **main diagonal** of the matrix.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

A **diagonal matrix** is a *square* matrix whose only nonzero entries are on the main diagonal.

$$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

The $n \times n$ **identity matrix** I_n is the diagonal matrix with all diagonal entries equal to 1. It is special because $I_n v = v$ for *all* v in $\mathbf{R}^n$.

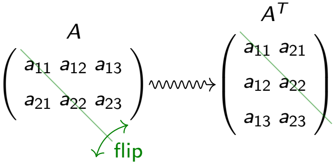
$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

More Notation for Matrices
Continued

The **zero matrix** (of size $m \times n$) is the $m \times n$ matrix 0 with all zero entries.

$$0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix A^T whose rows are the columns of A. In other words, the ij entry of A^T is a_{ji}.



Addition and Scalar Multiplication

You add two matrices component by component, like with vectors.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{pmatrix}$$

Note you can only add two matrices *of the same size*.

You multiply a matrix by a scalar by multiplying each component, like with vectors.

$$c \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} & ca_{13} \\ ca_{21} & ca_{22} & ca_{23} \end{pmatrix}.$$

These satisfy the expected rules, like with vectors:

$$\begin{aligned} A+B &= B+A & \quad (A+B)+C &= A+(B+C) \\ c(A+B) &= cA+cB & \quad (c+d)A &= cA+dA \\ (cd)A &= c(dA) & \quad A+0 &= A \end{aligned}$$

Matrix Multiplication

Beware: multiplication is more subtle than addition and scalar multiplication.

must be equal

Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix with columns $v_1, v_2 \ldots, v_p$:

$$B = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{pmatrix}.$$

The **product** AB is the $m \times p$ matrix with columns $Av_1, Av_2, \ldots, Av_p$:

$$AB \stackrel{\text{def}}{=} \begin{pmatrix} | & | & & | \\ Av_1 & Av_2 & \cdots & Av_p \\ | & | & & | \end{pmatrix}.$$

In order for $Av_1, Av_2, \ldots, Av_p$ to make sense, the number of columns of A has to be the same as the number of rows of B.

Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix} = \left(\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 14 & -10 \\ 32 & -28 \end{pmatrix}$$

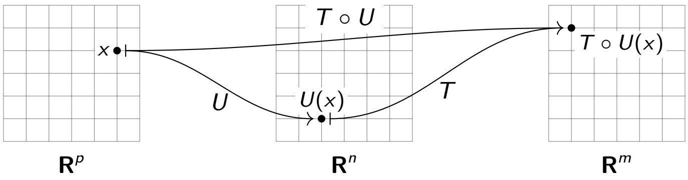
Why is this the correct definition of matrix multiplication?

Definition

Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ and $U\colon \mathbf{R}^p \to \mathbf{R}^n$ be transformations. The **composition** is the transformation

$$T \circ U\colon \mathbf{R}^p \to \mathbf{R}^m \quad \text{defined by} \quad T \circ U(x) = T(U(x)).$$

This makes sense because $U(x)$ (the output of U) is in $\mathbf{R}^n$, which is the domain of T (the inputs of T).



If T and U are linear then so is $T \circ U$. We have to check:

$$\begin{aligned} T \circ U(v + w) &= T(U(v + w)) = T(U(v) + U(w)) = T(U(v)) + T(U(w)) \\ &= T \circ U(v) + T \circ U(w) \end{aligned}$$

$$T \circ U(cv) = T(U(cv)) = T(cU(v)) = cT(U(v)) = cT \circ U(v)$$

Composition of Linear Transformations

Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ and $U\colon \mathbf{R}^p \to \mathbf{R}^n$ be *linear* transformations. Let A and B be their matrices:

$$A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix} \quad B = \begin{pmatrix} | & | & & | \\ U(e_1) & U(e_2) & \cdots & U(e_p) \\ | & | & & | \end{pmatrix}$$

Question

What is the matrix for $T \circ U$?

How do we find the matrix for $T \circ U$?

$$T \circ U(e_1) = T(U(e_1)) = T(Be_1) = A(Be_1) = (AB)e_1$$

because Be_1 is the first column of B, which is $U(e_1)$. For any other i, the same works:

$$T \circ U(e_i) = T(U(e_i)) = T(Be_i) = A(Be_i) = (AB)e_i.$$

This says that the ith column of the matrix for $T \circ U$ is the ith column of AB.

The matrix of the composition is the product of the matrices!

For example, let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix}.$$

Let $T(x) = Ax$ and $U(y) = By$, so

$$T\colon \mathbf{R}^3 \longrightarrow \mathbf{R}^2 \qquad U\colon \mathbf{R}^2 \longrightarrow \mathbf{R}^3 \qquad T \circ U\colon \mathbf{R}^2 \longrightarrow \mathbf{R}^2.$$

Let's find the matrix for $T \circ U$:

$$T \circ U(e_1) = T(U(e_1)) = T(Be_1) = T\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = A\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 14 \\ 32 \end{pmatrix}$$

$$T \circ U(e_2) = T(U(e_2)) = T(Be_2) = T\begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix} = A\begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} -10 \\ -28 \end{pmatrix}$$

Before we computed $AB = \begin{pmatrix} 14 & -10 \\ 32 & -28 \end{pmatrix}$, so AB is the matrix of $T \circ U$.

Poll

Do there exist *nonzero* matrices A and B with $AB = 0$?

Here's an example:

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \left(\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The Row-Column Rule for Matrix Multiplication

Recall: A row vector of length n times a column vector of length n is a scalar:

$$\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + \cdots + a_n b_n.$$

Another way of multiplying a matrix by a vector is:

$$Ax = \begin{pmatrix} \text{---} & r_1 & \text{---} \\ & \vdots & \\ \text{---} & r_m & \text{---} \end{pmatrix} x = \begin{pmatrix} r_1 x \\ \vdots \\ r_m x \end{pmatrix}.$$

On the other hand, you multiply two matrices by

$$AB = A \begin{pmatrix} | & & | \\ c_1 & \cdots & c_p \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ Ac_1 & \cdots & Ac_p \\ | & & | \end{pmatrix}.$$

It follows that

$$AB = \begin{pmatrix} \text{---} & r_1 & \text{---} \\ & \vdots & \\ \text{---} & r_m & \text{---} \end{pmatrix} \begin{pmatrix} | & & | \\ c_1 & \cdots & c_p \\ | & & | \end{pmatrix} = \begin{pmatrix} r_1 c_1 & r_1 c_2 & \cdots & r_1 c_p \\ r_2 c_1 & r_2 c_2 & \cdots & r_2 c_p \\ \vdots & \vdots & & \vdots \\ r_m c_1 & r_m c_2 & \cdots & r_m c_p \end{pmatrix}$$

The Row-Column Rule for Matrix Multiplication

The ij entry of $C = AB$ is the ith row of A times the jth column of B:

$$c_{ij} = (AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

This is how everybody on the planet actually computes AB. Diagram ($C = AB$):

$$\begin{pmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ik} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mk} & \cdots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{k1} & \cdots & b_{kj} & \cdots & b_{kp} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mp} \end{pmatrix}$$

ith row — jth column — ij entry

Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 2\cdot 2 + 3\cdot 3 & \square \\ \square & \square \end{pmatrix} = \begin{pmatrix} 14 & \square \\ \square & \square \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} \square & \square \\ 4\cdot 1 + 5\cdot 2 + 6\cdot 3 & \square \end{pmatrix} = \begin{pmatrix} \square & \square \\ 32 & \square \end{pmatrix}$$

Properties of Matrix Multiplication

Mostly matrix multiplication works like you'd expect. Suppose A has size $m \times n$, and that the other matrices below have the right size to make multiplication work.

$$
\begin{aligned}
A(BC) &= (AB)C & \qquad A(B+C) &= AB + AC \\
(B+C)A &= BA + CA & c(AB) &= (cA)B \\
c(AB) &= A(cB) & I_n A &= A \\
AI_m &= A
\end{aligned}
$$

Most of these are easy to verify. For instance, $I_n A = A$ because $I_n v = v$ for every vector, and multiplication works column-by-column. *Associativity* is $A(BC) = (AB)C$. It is a pain to verify using the row-column rule! Much easier: note that for transformations S, T, U, one has

$$
\begin{aligned}
S \circ (T \circ U)(x) &= S(T \circ U(x)) = S(T(U(x))) \\
&= (S \circ T)(U(x)) = (S \circ T) \circ U(x).
\end{aligned}
$$

In other words, matrix multiplication is associative *because* composition of transformations is (obviously) associative.

This is a good example of an instance where having a conceptual viewpoint saves you a lot of work!

Warning!

- AB is usually not equal to BA.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$$

  In fact, BA may not even be defined.

- $AB = AC$ does not imply $B = C$, even if $A \neq 0$.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix}$$

- $AB = 0$ does not imply $A = 0$ or $B = 0$.

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Other Reading

Read about powers of a matrix and multiplication of transposes in §2.1.