Better Reasoning (Question I worksheet 4)

①

If v_1, v_2, v_3 span $\mathbb{R}^3$ then the matrix $A = (v_1, v_2, v_3)$ must have a pivot in each row. If not (ex) $\left(\begin{array}{ccc|c} 1 & 0 & 0 & b_1 \\ 0 & 0 & 0 & b_2 \\ 0 & 0 & 0 & b_3 \end{array}\right)$ then there is a value of b_2 or b_3 that makes this inconsistant which means the columns would not span all of $\mathbb{R}^3$

If A has a pivot in every row & we have 3 vectors in $\mathbb{R}^3$ then the matrix A when row reduced looks like this $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Since A is square & has a pivot in every row, it must have a pivot in every column. This means that the columns of A are linearly independant (We discussed this in class).

There fore, $\{v_1, v_2, v_3\}$ must be linearly independant if they span $\mathbb{R}^3$.

Major points

given vectors $\{v_1, v_2, \dots, v_n\} \in \mathbb{R}^m$

☆ if $A = (v_1, v_2, \dots, v_n)$ has a pivot in every row, then $\{v_1, v_2, \dots, v_n\}$ span $\mathbb{R}^m$.

↳ only if $m \leq n$

$A_{m\times n} = {}_{m}\overset{n}{\square}$

☆ if $A = (v_1, v_2, \dots, v_n)$ has a pivot in every column, then $\{v_1, v_2, \dots, v_3\}$ are linearly independant

↳ only if $m \geq n$.

$A_{m\times n} = {}_{m}\overset{n}{\square}$

Math 1553 Worksheet 5

September 23, 2016

Linear Transformations

$u \in \mathbb{R}^n \quad v \in \mathbb{R}^m$

Remember: $T(u) = v \Rightarrow \underset{(m \times n)}{A} \cdot u = v$

1. Which of the following transformations are onto? Which are one-to-one? If the transformation is not onto, find a vector not in the range. If the matrix is not one-to-one, find two vectors with the same image.

a) Counterclockwise rotation by 32° in $\mathbf{R}^2$.

$A = \begin{pmatrix} \cos 32^\circ & -\sin 32^\circ \\ \sin 32^\circ & \cos 32^\circ \end{pmatrix}$

1-1 - each vector gets rotated to another one

onto - codomain is $\mathbb{R}^2$ and domain is $\mathbb{R}^2$ / rotating 2 lin indep. vectors gives us 2 more

b) The transformation $T: \mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(x,y,z) = (z,x)$.

$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

not 1-1 : $T(1,0,1) = (1,1)$ & $T(1,20,1) = (1,1)$

onto : can pick any x or z, so all points in $\mathbb{R}^2$ are covered in this transformation

c) The transformation $T: \mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(x,y,z) = (0,x)$.

$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

not 1-1 : $T(1,0,1) = (0,1)$ & $T(1,20,1) = (0,1)$

not onto: (codomain is $\mathbb{R}^2$ but $(0,x)$ can only be a line tracking change in only x values.

(ex) $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ not in range

d) The matrix transformation with matrix $A = \begin{pmatrix} 1 & 6 \\ -1 & 2 \\ 2 & -1 \end{pmatrix}$.

$\begin{pmatrix} 1 & 6 \\ -1 & 2 \\ 2 & -1 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow T: \mathbb{R}^2 \to \mathbb{R}^3$

1-1 - all vectors in $\mathbb{R}^2$ in domain go to unique vectors in $\mathbb{R}^3$

not onto - can't get $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ for example / any value

e) The matrix transformation with matrix $A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

$T: \mathbb{R}^3 \to \mathbb{R}^3$

not 1-1 : $v_1 = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$; $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ → $Av_1 = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = Av_2$ $\quad v_1 \& v_2 \to \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$

not onto : can't ever get $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ (don't have pivot in every row)

2. Say A is an $m \times 2$ matrix. Let $T: \mathbf{R}^2 \to \mathbf{R}^m$ be the transformation defined by $T(x) = Ax$. If the columns of A are linearly independent, what does the image of T look like geometrically? What if they're linearly dependent?

$A = m \overset{2}{\begin{pmatrix} x & x \\ x & x \\ \vdots & \\ x & x \end{pmatrix}}$

a) if columns are lin indep :

$\xrightarrow{RREF} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}$ 1-1 not onto

image looks like a plane in $\mathbb{R}^m$ (can only uniquely define

b) if colums are lin dep.

$\xrightarrow{RREF} \begin{pmatrix} 1 & x \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}$

3. For each matrix, describe what the associated matrix transformation does to $\mathbf{R}^3$ geometrically.

a) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **b)** $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ **c)** $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

a)
- Swaps x & y coords.
- maps to all of $\mathbb{R}^3$

Image = $\mathbb{R}^3$

b)
- Swaps x & y coordinates
- projects all points onto xy plane.

Image = $\mathbb{R}^2$

c)
- projects all points onto the z axis.

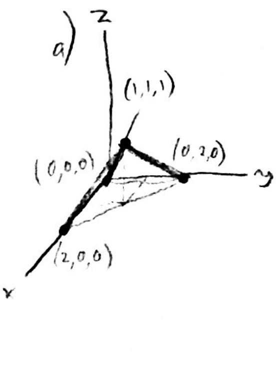
Image = $\mathbb{R}^1$

4. The second little pig has decided to build his house out of sticks. His house is shaped like a pyramid with a triangular base that has vertices at the points $(0, 0, 0)$, $(2, 0, 0)$, $(0, 2, 0)$, and $(1, 1, 1)$. The big bad wolf finds the pig's house and blows it down so that the house is rotated by an angle of 45° in a counterclockwise direction about the z-axis, and then projected onto the xy-plane.

a) Express this transformation as a composition of two simpler transformations.

b) Find a matrix C to represent the transformation that destroys the house. Express this as a product of the two matrices corresponding to the transformations in (a).

a)



first

rotation around z-axis by 45°

$$\begin{pmatrix} \cos 45 & -\sin 45 & 0 \\ \sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- z coordinates stays the same,
- x & y coordinates change based on rotation matrix.

second.

Project onto xy plane

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- eq to making z term 0.

b)

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \cos 45 & -\sin 45 & 0 \\ \sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv \overset{C}{\boxed{\begin{pmatrix} \cos 45 & -\sin 45 & 0 \\ \sin 45 & \cos 45 & 0 \\ 0 & 0 & 0 \end{pmatrix}}} \overset{ex}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} = \begin{pmatrix} \cos 45 - \sin 45 \\ \sin 45 + \cos 45 \\ 0 \end{pmatrix}$$

$\Updownarrow$

check point $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. first: $\begin{pmatrix} \cos 45 & -\sin 45 & 0 \\ \sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos 45 - \sin 45 \\ \sin 45 + \cos 45 \\ 1 \end{pmatrix}$ second $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos 45 - \sin 45 \\ \sin 45 + \cos 45 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos 45 - \sin \\ \sin 45 + \cos \\ 0 \end{pmatrix}$