Announcements

September 26

- The first midterm is this Friday, during recitation. It covers Chapter 1, sections 1–5 and 7–9.
  - About half the problems will be conceptual, and the other half computational.
- Homeworks 1.9 and 2.1 are also due Friday, *early in the morning*.
- There is a practice midterm posted on the website. It is identical in format to the real midterm (although there may be $\pm$1–2 problems).
- Study tips:
  - There are lots of problems at the end of each section in the book, and at the end of the chapter, for practice.
  - Make sure to learn the theorems and learn the definitions, and understand what they mean.
  - Sit down to do the practice midterm in 50 minutes, with no notes.
  - Come to office hours!
- **Triple office hours:** today, 3–5pm; Wednesday, 1–3pm; Thursday, 2:30–4:30pm.
  - As always, TAs' office hours are posted on the website.
  - Math Lab is also a good place to visit.

Section 2.2

The Inverse of a Matrix

The Definition of Inverse

The multiplicative inverse (or reciprocal) of a number a is the number b such that $ab = 1$. Since multiplication of numbers is commutative, this is the same as saying $ba = 1$. We define the inverse of a matrix the same way, but we have to require *both*.

Definition

Let A be an $n \times n$ square matrix. We say A is **invertible** (or **nonsingular**) if there is a matrix B of the same size, such that

$$AB = I_n \quad \text{and} \quad BA = I_n.$$

identity matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

In this case, B is the **inverse** of A, and is written A^{-1}.

Example

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

I claim $B = A^{-1}$. Check:

$$AB = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The 2×2 case

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The **determinant** of A is the number

$$\boxed{\det(A) = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.}$$

Facts:

1. If $\det(A) \neq 0$, then A is invertible and $\boxed{A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.}$
2. If $\det(A) = 0$, then A is not invertible.

Why 1?

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

So we get the identity by dividing by $ad - bc$.

Why 2? Later, when we talk about determinants.

For example,

$$\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1 \cdot 4 - 2 \cdot 3 = -2 \qquad \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = -\frac{1}{2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}.$$

Fact: if A is invertible, then $Ax = b$ has exactly one solution for every b:

$$x = A^{-1}b.$$

Why?

$$Ax = b \rightsquigarrow A^{-1}(Ax) = A^{-1}b \rightsquigarrow (A^{-1}A)x = A^{-1}b$$

$I_n x = x$ for every x

$$\rightsquigarrow I_n x = A^{-1}b \rightsquigarrow x = A^{-1}b.$$

Example

Solve the system

could be any other vector

$$\begin{aligned} 2x + 3y + 2z &= \boxed{1} \\ x \qquad\quad + 3z &= \boxed{1} \\ 2x + 2y + 3z &= \boxed{1} \end{aligned} \quad \text{using} \quad \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}.$$

Answer: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} \boxed{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$

Some Facts

Say A and B are invertible $n \times n$ matrices.

1. A^{-1} is invertible and its inverse is $(A^{-1})^{-1} = A$.
2. AB is invertible and its inverse is $(AB)^{-1} = \cancel{A^{-1}B^{-1}}\ B^{-1}A^{-1}$.

   Why? $(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n$.
3. A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

   Why? $A^T(A^{-1})^T = (A^{-1}A)^T = I_n^T = I_n$.

Poll

If A, B, C are invertible $n \times n$ matrices, what is the inverse of ABC?

i. $A^{-1}B^{-1}C^{-1}$ ii. $B^{-1}A^{-1}C^{-1}$ iii. $C^{-1}B^{-1}A^{-1}$ iv. $C^{-1}A^{-1}B^{-1}$

It's (iii):

$$\begin{aligned}(ABC)(C^{-1}B^{-1}A^{-1}) &= AB(CC^{-1})B^{-1}A^{-1} = A(BB^{-1})A^{-1} \\ &= AA^{-1} = I_n.\end{aligned}$$

In general, a product of invertible matrices is invertible, and the inverse is the product of the inverses, in the reverse order.

Computing A^{-1}

Let A be an $n \times n$ matrix. Here's how to compute A^{-1}.

1. Row reduce the augmented matrix $(\, A \mid I_n \,)$.
2. If the result has the form $(\, I_n \mid B \,)$, then A is invertible and $B = A^{-1}$.
3. Otherwise, A is not invertible.

Example

Let's compute $\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix}^{-1}$.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 3 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -6 & -2 \\ 0 & 1 & 0 & 0 & -2 & -1 \\ 0 & 0 & 2 & 0 & 3 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -6 & -2 \\ 0 & 1 & 0 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 & 3/2 & 1/2 \end{array}\right)$$

So $\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -6 & -2 \\ 0 & -2 & -1 \\ 0 & 3/2 & 1/2 \end{pmatrix}$.

Why Does This Work?

First answer: we can think of the algorithm as simultaneously solving the equations

$$Ax_1 = e_1 : \qquad \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array}\right)$$

$$Ax_2 = e_2 : \qquad \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array}\right)$$

$$Ax_3 = e_3 : \qquad \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array}\right)$$

Now note $A^{-1}e_i = A^{-1}(Ax_i) = x_i$, and x_i is the ith column in the augmented part. Also $A^{-1}e_i$ is the ith column of A^{-1}.

Second answer: elementary matrices.

Definition

An **elementary matrix** is a square matrix E which differs from I_n by one row operation.

There are three kinds:

scaling	adding	swapping
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Fact: if E is the elementary matrix for a row operation, then EA differs from A by the same row operation.

Why? Check for each type. For example:

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 4 \\ 2 \cdot 1 + 0 & 2 \cdot 0 + 1 & 2 \cdot 4 + 2 \\ 0 & -3 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 10 \\ 0 & -3 & -4 \end{pmatrix} \qquad (R_2 = R_2 + 2R_1)$$

Fact: if E is the elementary matrix for a row operation, then EA differs from A by the same row operation.

So if you first do a row operation to I_n to get E, then you do the opposite row operation to I_n to get E', then $E'E = I_n$ (multiplication by E' undoes what you did to I_n to get E).

It follows that *elementary matrices are invertible* and that *the inverse is the elementary matrix which does the opposite row operation*.

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{scale } R_2 \text{ by } 2}^{-1} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{scale } R_2 \text{ by } 1/2} \qquad \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{add } 2R_1 \text{ to } R_2}^{-1} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{subtract } 2R_1 \text{ from } R_2}$$

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{swap } R_1 \text{ and } R_2}^{-1} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{swap } R_1 \text{ and } R_2}$$

Theorem

An $n \times n$ matrix A is invertible if and only if it is row equivalent to I_n. In this case, the sequence of row operations taking A to I_n also takes I_n to A^{-1}.

Why? Say the row operations taking A to I_n have elementary matrices $E_1, E_2, \ldots, E_k$. So

$$\text{note the order!} \longrightarrow E_k E_{k-1} \cdots E_2 E_1 A = I_n$$
$$\implies E_k E_{k-1} \cdots E_2 E_1 A A^{-1} = A^{-1}$$
$$\implies E_k E_{k-1} \cdots E_2 E_1 I_n = A^{-1}.$$

This means if you do these same row operations to A and to I_n, you'll end up with I_n and A^{-1}. This is what you do when you row reduce the augmented matrix:

$$\left(A \mid I_n \right) \rightsquigarrow \left(I_n \mid A^{-1} \right).$$

Section 2.3

Characterization of Invertible Matrices

Invertible Transformations

Definition

A transformation $T\colon \mathbf{R}^n \to \mathbf{R}^n$ is **invertible** if there exists another transformation $U\colon \mathbf{R}^n \to \mathbf{R}^n$ such that

$$T \circ U(x) = x \quad \text{and} \quad U \circ T(x) = x$$

for all x in $\mathbf{R}^n$. In this case we say U is the **inverse** of T, and we write $U = T^{-1}$.

In other words, $T(U(x)) = x$, so T "undoes" U, and likewise U "undoes" T.

Fact

A transformation T is invertible if and only if it is both one-to-one and onto.

If T is one-to-one and onto, this means for every y in $\mathbf{R}^n$, there is a unique x in $\mathbf{R}^n$ such that $T(x) = y$. Then $T^{-1}(y) = x$.

Invertible Linear Transformations

If $T\colon \mathbf{R}^n \to \mathbf{R}^n$ is an invertible *linear* transformation with matrix A, then what is the matrix for T^{-1}? Let's call it B. We know $T \circ T^{-1}$ has matrix AB, so for all x,

$$ABx = T \circ T^{-1}(x) = x.$$

Hence $AB = I_n$, so $B = A^{-1}$.

Fact

If T is an invertible linear transformation with matrix A, then T^{-1} is an invertible linear transformation with matrix A^{-1}.

The Invertible Matrix Theorem

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix, and let $T\colon \mathbf{R}^n \to \mathbf{R}^n$ be the linear transformation $T(x) = Ax$. The following statements are equivalent.

1. A is invertible.
2. T is invertible.
3. A is row equivalent to I_n.
4. A has n pivots.
5. $Ax = 0$ has only the trivial solution.
6. The columns of A are linearly independent.
7. T is one-to-one.
8. $Ax = b$ is consistent for all b in $\mathbf{R}^n$.
9. The columns of A span $\mathbf{R}^n$.
10. T is onto.
11. A has a left inverse (there exists B such that $BA = I_n$).
12. A has a right inverse (there exists B such that $AB = I_n$).
13. A^T is invertible.

you really have to know these

There are two kinds of *square* matrices:

1. invertible (non-singular), and
2. non-invertible (singular).

For invertible matrices, all statements of the Invertible Matrix Theorem are true.

For non-invertible matrices, all statements of the Invertible Matrix Theorem are false.

I *highly recommend* going through them and trying to figure out on your own (or at least with help from the book) why they're all equivalent.

You know enough at this point to be able to reduce all of the statements to assertions about the pivots of a square matrix.