Announcements
October 3

- The midterm will be returned in recitation on Friday.
- Keep tabs on your grades in T-Square.
- Homeworks 2.2 and 2.3 are due Friday at 6am.
- Quiz on Friday: 2.1, 2.2, 2.3.
- Office hours: Wednesday 1–2pm, Thursday 2:30–4:30pm.
  - As always, TAs' office hours are posted on the website.
  - Math Lab is also a good place to visit.

Section 2.8

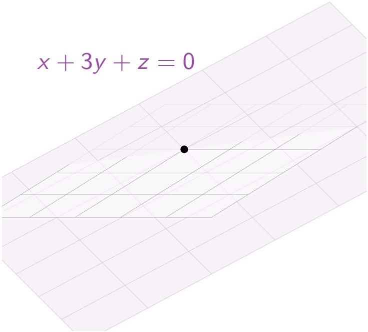
Subspaces of $\mathbf{R}^n$

Motivation

Today we will discuss **subspaces** of $\mathbf{R}^n$.

A subspace turns out to be the same as a span, except we don't care which vectors it's the span of.

This arises naturally when you have, say, a plane through the origin in $\mathbf{R}^3$ which is *not* defined (a priori) as a span, but you still want to say something about it.



Definition of Subspace

Definition

A **subspace** of $\mathbf{R}^n$ is a subset V of $\mathbf{R}^n$ satisfying:

1. The zero vector is in V. "not empty"
2. If u and v are in V, then $u + v$ is also in V. "closed under addition"
3. If u is in V and c is in $\mathbf{R}$, then cu is in V. "closed under $\times$ scalars"

What does this mean?

- If v is in V, then all scalar multiples of v are in V by (3). That is, the line through v is in V.
- If u, v are in V, then xu and yv are in V for scalars x, y by (3). So $xu + yv$ is in V by (2). So $\text{Span}\{u, v\}$ is contained in V.
- Likewise, if $v_1, v_2, \ldots, v_n$ are all in V, then $\text{Span}\{v_1, v_2, \ldots, v_n\}$ is contained in V.

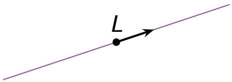
Summary: a subspace V has the property that it contains the span of any set of vectors in V.

In particular, V *is a span*: it is the span of all of the vectors in V. (We'll find a better spanning set later.)

Examples

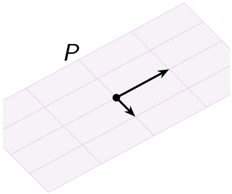
Example

A line L through the origin: this contains the span of any vector in L.



Example

A plane P through the origin: this contains the span of any vectors in P.



Example

All of $\mathbf{R}^n$: this contains 0, and is closed under addition and scalar multiplication.

Example

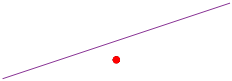
The subset $\{0\}$.

Note these are all pictures of spans! (Line, plane, space, etc.)

Non-Examples

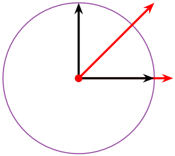
Non-Example

A line L (or any other set) that doesn't contain the origin is not a subspace. Fails: 1.



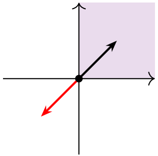
Non-Example

A circle C is not a subspace. Fails: 1,2,3. Think: a circle isn't a "linear space."



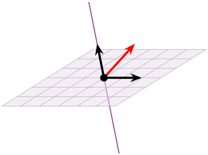
Non-Example

The first quadrant in $\mathbf{R}^2$ is not a subspace. Fails: 3 only.



Non-Example

A line union a plane in $\mathbf{R}^3$ is not a subspace. Fails: 2 only.



Spans are Subspaces

Fact: any $\mathrm{Span}\{v_1, v_2, \ldots, v_n\}$ is a subspace.

Check:

1. $0 = 0v_1 + 0v_2 + \cdots + 0v_n$ is in the span.
2. If, say, $u = 3v_1 + 4v_2$ and $v = -v_1 - 2v_2$, then

$$u + v = 3v_1 + 4v_2 - v_1 - 2v_2 = 2v_1 + 2v_2$$

   is also in the span.
3. Similarly, if u is in the span, then so is cu for any scalar c.

Every subspace is a span, and every span is a subspace.

Definition

If $V = \mathrm{Span}\{v_1, v_2, \ldots, v_n\}$, we say that V is the subspace **generated by** or **spanned by** the vectors $v_1, v_2, \ldots, v_n$.

Poll

Which are subspaces? For those that are not, which properties do they fail?

A. $\left\{\begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbf{R}^2 \mid a = 0\right\}$

B. $\left\{\begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbf{R}^2 \mid a + b = 0\right\}$

C. $\left\{\begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbf{R}^2 \mid ab = 0\right\}$

D. $\left\{\begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbf{R}^2 \mid ab \neq 0\right\}$

E. $\left\{\begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbf{R}^2 \mid a, b \text{ are rational}\right\}$

Column Space and Null Space

Let A be an $m \times n$ matrix. It naturally gives rise to *two* subspaces.

Definition

- The **column space** of A is the subspace of $\mathbf{R}^m$ spanned by the columns of A. It is written $\operatorname{Col} A$.
- The **null space** of A is the set of all solutions of the homogeneous equation $Ax = 0$:
$$\operatorname{Nul} A = \{x \text{ in } \mathbf{R}^n \mid Ax = 0\}.$$
This is a subspace of $\mathbf{R}^n$.

The column space is defined as a span, so we know it is a subspace. It is the range (as opposed to the codomain) of the transformation $T(x) = Ax$.

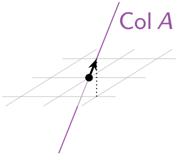
Check that the null space is a subspace:

1. 0 is in $\operatorname{Nul} A$ because $A0 = 0$.
2. If u and v are in $\operatorname{Nul} A$, then $Au = 0$ and $Av = 0$. Hence
$$A(u + v) = Au + Av = 0,$$
so $u + v$ is in $\operatorname{Nul} A$.
3. If u is in $\operatorname{Nul} A$, then $Au = 0$. For any scalar c, $A(cu) = cAu = 0$. So cu is in $\operatorname{Nul} A$.

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Let's compute the column space:

$$\text{Col}\, A = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$



This is a line in $\mathbf{R}^3$.

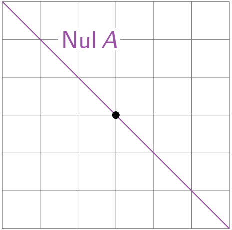
Let's compute the null space:

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x+y \\ x+y \end{pmatrix}.$$

This zero if and only if $x = -y$. So

$$\text{Nul}\, A = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \mid y = -x \right\}.$$

This defines a line in $\mathbf{R}^2$:



The Null Space is a Span

The column space of a matrix A is defined to be a span (of the columns).

The null space is defined to be the solution set to $Ax = 0$. It is a subspace, so it is a span.

Question

How to find vectors which span the null space?

Answer: parametric vector form! We know that the solution set to $Ax = 0$ has a parametric form that looks like

$$x_3 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

if, say, x_3 and x_4 are the free variables. So

$$\operatorname{Nul} A = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Refer back to the slides for 9/12 (Solution Sets).

Note: It is much easier to define the null space first as a subspace, then find spanning vectors *later*, if we need them. This is one reason subspaces are so useful.

The Null Space is a Span

Example, Revisited

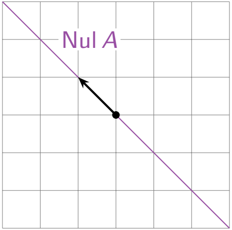
Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$. The reduced row echelon form is $\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$.

This gives the equation $x + y = 0$, or

$$\begin{aligned} x &= -y \\ y &= y. \end{aligned} \quad \xrightarrow{\text{parametric vector form}} \quad \begin{pmatrix} x \\ y \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

The null space is

$$\operatorname{Nul} A = \operatorname{Span}\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}.$$



Basis of a Subspace

How many vectors are needed to span a given subspace?

Definition

Let V be a subspace of $\mathbf{R}^n$. A **basis** of V is a set of vectors $\{v_1, v_2, \ldots, v_m\}$ in $\mathbf{R}^n$ such that:

1. $V = \mathrm{Span}\{v_1, v_2, \ldots, v_m\}$, and
2. $\{v_1, v_2, \ldots, v_m\}$ is linearly independent.

The number of vectors in a basis is the **dimension** of V, and is written $\dim V$.

Note the big red border here

Why is a basis the smallest number of vectors needed to span?

Recall: *linearly independent* means that every time you add another vector, the span gets bigger.

Hence, if we remove any vector, the span gets *smaller*: so any smaller set can't span V.

Important

A subspace has *many different* bases, but they all have the same number of vectors (see the exercises in §2.9).

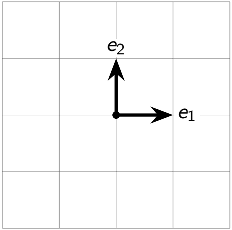
Bases of $\mathbf{R}^2$

Question

What is a basis for $\mathbf{R}^2$?

We need two vectors that *span* $\mathbf{R}^2$ and are *linearly independent*. $\{e_1, e_2\}$ is one basis.

1. They span: $\binom{a}{b} = ae_1 + be_2$.
2. They are linearly independent because they are not collinear.

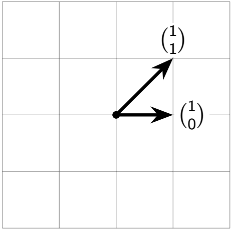


Question

What is another basis for $\mathbf{R}^2$?

Any two nonzero vectors that are not collinear. $\left\{\binom{1}{0}, \binom{1}{1}\right\}$ is also a basis.

1. They span: $\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]$ has a pivot in every row.
2. They are linearly independent: $\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]$ has a pivot in every column.



Bases of $\mathbf{R}^n$

The unit coordinate vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \ldots, \quad e_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

are a basis for $\mathbf{R}^n$. The identity matrix has columns $e_1, e_2, \ldots, e_n$.

1. They span: I_n has a pivot in every row.
2. They are linearly independent: I_n has a pivot in every column.

In general: $\{v_1, v_2, \ldots, v_n\}$ is a basis for $\mathbf{R}^n$ if and only if the matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$$

has a pivot in every row and every column, i.e. if A is *invertible*.

Basis for Nul A

Fact

The vectors in the parametric vector form of the general solution to $Ax = 0$ always form a basis for Nul A.

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{parametric vector}} x = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{\text{basis of Nul } A} \left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

1. The vectors span Nul A by construction (every solution to $Ax = 0$ has this form).
2. Can you see why they are linearly independent? (Look at the last two rows.)

Basis for Col A

Fact

The *pivot columns* of A always form a basis for Col A.

Warning: I mean the pivot columns of the *original* matrix A, not the row-reduced form. (Row reduction changes the column space.)

Example

$$A = \begin{pmatrix} \boxed{\begin{matrix} 1 \\ -2 \\ 2 \end{matrix}} & \boxed{\begin{matrix} 2 \\ -3 \\ 4 \end{matrix}} & \begin{matrix} 0 \\ 4 \\ 0 \end{matrix} & \begin{matrix} -1 \\ 5 \\ -2 \end{matrix} \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot columns = basis $\leftarrow$ pivot columns in rref

So a basis for Col A is

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \right\}.$$

Why? End of §2.8, or ask in office hours.