- ▶ The midterm will be returned in recitation on Friday.

- ▶ Homeworks 2.2 and 2.3 are due Friday at 6am.

- ▶ Quiz on Friday: 2.1, 2.2, 2.3.

- ▶ Midterm 2 will take place in Recitation on Friday, 10/28.

- ▶ Office hours: today 1–2pm, tomorrow 3:30–4:30pm, and by appointment, in Skiles 221.
  - ▶ As always, TAs' office hours are posted on the website.
  - ▶ Math Lab is also a good place to visit.

Section 2.9

Dimension and Rank

Coefficients of Basis Vectors

Recall: a **basis** of a subspace V is a set of vectors that *spans* V and is *linearly independent*.

Lemma ← ──── like a theorem, but less important

If $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ is a basis for a subspace V, then any vector x in V can be written as a linear combination

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$$

for *unique* coefficients $c_1, c_2, \ldots, c_m$.

We know x is a linear combination of the v_i because they span V. Suppose that we can write x as a linear combination with different coefficients:

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$$
$$x = c_1' v_1 + c_2' v_2 + \cdots + c_m' v_m$$

Subtracting:

$$0 = x - x = (c_1 - c_1')v_1 + (c_2 - c_2')v_2 + \cdots + (c_m - c_m')v_m$$

Since $v_1, v_2, \ldots, v_m$ are linearly independent, they only have the trivial linear dependence relation. That means each $c_i - c_i' = 0$, or $c_i = c_i'$.

The unit coordinate vectors $e_1, e_2, \ldots, e_n$ form a basis for $\mathbf{R}^n$. Any vector is a unique linear combination of the e_i:

$$v = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 5 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 3e_1 + 5e_2 - 2e_3.$$

Observe: the *coordinates* of v are exactly the *coefficients* of e_1, e_2, e_3.

We can go backwards: given any basis $\mathcal{B}$, we interpret the coefficients of a linear combination as "coordinates" with respect to $\mathcal{B}$.

Definition
Let $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ be a basis of a subspace V. Any vector x in V can be written uniquely as a linear combination $x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$. The coefficients $c_1, c_2, \ldots, c_m$ are the **coordinates of x with respect to $\mathcal{B}$**. The $\mathcal{B}$-**coordinate vector of x** is the vector

$$[x]_\mathcal{B} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} \quad \text{in } \mathbf{R}^m.$$

Let $v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathcal{B} = \{v_1, v_2\}$, $V = \text{Span}\{v_1, v_2\}$.

Verify that $\mathcal{B}$ is a basis:

Span: by definition $V = \text{Span}\{v_1, v_2\}$.

Linearly independent: because they are not multiples of each other.

Question: If $[x]_{\mathcal{B}} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$, then what is x?

$$[x]_{\mathcal{B}} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \quad \text{means} \quad x = 5v_1 + 2v_2 = 5\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \\ 7 \end{pmatrix}.$$

Question: Find the $\mathcal{B}$-coordinates of $x = \begin{pmatrix} 5 \\ 3 \\ 5 \end{pmatrix}$.

We have to solve the vector equation $x = c_1 v_1 + c_2 v_2$ in the unknowns c_1, c_2.

$$\begin{pmatrix} 1 & 1 & | & 5 \\ 0 & 1 & | & 3 \\ 1 & 1 & | & 5 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & | & 5 \\ 0 & 1 & | & 3 \\ 0 & 0 & | & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & 3 \\ 0 & 0 & | & 0 \end{pmatrix}$$

So $x = 2v_1 + 3v_2$ and $[x]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Let $v_1 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 2 \\ 8 \\ 6 \end{pmatrix}$, $V = \text{Span}\{v_1, v_2, v_3\}$.

Question: find a basis for V.
V is the column span of the matrix

$$A = \begin{pmatrix} 2 & -1 & 2 \\ 3 & 1 & 8 \\ 2 & 1 & 6 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

A basis for the column span is formed by the pivot columns: $\mathcal{B} = \{v_1, v_2\}$.

Question: Find the $\mathcal{B}$-coordinates of $x = \begin{pmatrix} 4 \\ 11 \\ 8 \end{pmatrix}$.

We have to solve $x = c_1 v_1 + c_2 v_2$.

$$\begin{pmatrix} 2 & -1 & | & 4 \\ 3 & 1 & | & 11 \\ 2 & 1 & | & 8 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & | & 3 \\ 0 & 1 & | & 2 \\ 0 & 0 & | & 0 \end{pmatrix}$$

So $x = 3v_1 + 2v_2$ and $[x]_\mathcal{B} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Bases as Coordinate Systems

If $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ is a basis for a subspace V and x is in V, then finding the $\mathcal{B}$-coordinates for x means solving the vector equation

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$$

in the unknowns $c_1, c_2, \ldots, c_m$. These are the $\mathcal{B}$-coordinates. This (usually) means row reducing the augmented matrix

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_m & x \\ | & | & & | & | \end{array} \right).$$

Question: what happens if you try to find the $\mathcal{B}$-coordinates of x *not* in V? You end up with an inconsistent system: V is the span of $v_1, v_2, \ldots, v_m$, and if x is not in the span, then $x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$ has no solution.

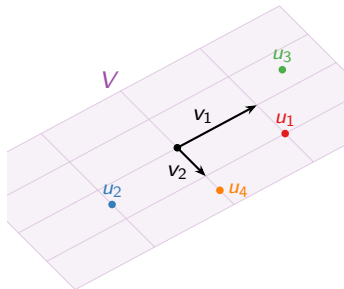# Bases as Coordinate Systems
Picture

Let

$$v_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

These form a basis $\mathcal{B}$ for the plane

$$V = \text{Span}\{v_1, v_2\}$$

in $\mathbf{R}^3$.

Question: estimate the $\mathcal{B}$-coordinates of these vectors:

$$[u_1]_\mathcal{B} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad [u_2]_\mathcal{B} = \begin{pmatrix} -1 \\ \frac{1}{2} \end{pmatrix} \quad [u_3]_\mathcal{B} = \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{pmatrix} \quad [u_4]_\mathcal{B} = \begin{pmatrix} 0 \\ \frac{3}{2} \end{pmatrix}$$

Remark
Many of you want to think of a plane in $\mathbf{R}^3$ as "being" $\mathbf{R}^2$. Choosing a basis $\mathcal{B}$ and using $\mathcal{B}$-coordinates is one way to make sense of that. But remember that the coordinates are the coefficients of a linear combination of the basis vectors.

The Rank Theorem

- The **dimension** of a subspace V is the number of vectors in a basis for V.
- A basis for the column space of a matrix A is given by the pivot columns.
- A basis for the null space of A is given by the vectors attached to the free variables in the parametric vector form.

Definition
The **rank** of a matrix A, written rank A, is the dimension of the column space Col A.

Observe:
$$\text{rank } A = \dim \text{Col } A = \text{the number of columns with pivots}$$
$$\dim \text{Nul } A = \text{the number of free variables}$$
$$= \text{the number of columns without pivots.}$$

Rank Theorem
If A is an $m \times n$ matrix, then

$$\text{rank } A + \dim \text{Nul } A = n = \text{the number of columns of } A.$$

Rank Theorem

If A is an $m \times n$ matrix, then

$$\text{rank } A + \dim \text{Nul } A = n = \text{the number of columns of } A.$$

What does this mean? In the equation $Ax = b$,

▶ You have some number of degrees of freedom in choosing b for which $Ax = b$ is consistent (the column span).

▶ For a given b in the column span, you have some number of degrees of freedom in choosing x (the solution set).

▶ These two numbers always sum to n.

This is a nontrivial relationship between the *solution set of $Ax = b$* and the *space of all b such that $Ax = b$ is consistent*.
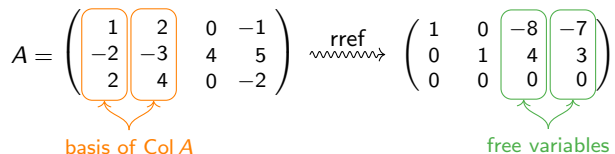
Example

If $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, then rank $A = 1$ and dim Nul $A = 2 = 3 - 1$.

$$A = \begin{pmatrix} \boxed{\begin{matrix} 1 \\ -2 \\ 2 \end{matrix}} & \boxed{\begin{matrix} 2 \\ -3 \\ 4 \end{matrix}} & \begin{matrix} 0 \\ 4 \\ 0 \end{matrix} & \begin{matrix} -1 \\ 5 \\ -2 \end{matrix} \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & \boxed{-8} & \boxed{-7} \\ 0 & 1 & \boxed{4} & \boxed{3} \\ 0 & 0 & \boxed{0} & \boxed{0} \end{pmatrix}$$

basis of Col A free variables

Since the first two columns are a basis for Col A, the rank is 2, and any b in Col A can be written uniquely as

$$b = c_1 \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}.$$

So there are two degrees of freedom in choosing the possible b's.

Since there are two free variables x_3, x_4, any solution to $Ax = b$ (for b in Col A) can be written uniquely in vector parametric form as

$$x = p + x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix},$$

where p is a particular solution. There are two degrees of freedom in choosing x. The Rank Theorem says $2 + 2 = 4$.

Let A and B be 3×3 matrices. Suppose that rank$(A) =$ 2 and rank$(B) = 2$. Is it possible that $AB = 0$? Why or why not?

If $AB = 0$, then $ABx = 0$ for every x in $\mathbf{R}^3$.

This means $A(Bx) = 0$, so Bx is in Nul A.

This is true for every x, so Col B is contained in Nul A.

But dim Nul $A = 1$ and dim Col $B = 2$, and a 1-dimensional space can't contain a 2-dimensional space.

Hence it can't happen.

Basis Theorem
Let V be a subspace of dimension m. Then:

- Any m linearly independent vectors in V form a basis for V.
- Any m vectors that span V form a basis for V.

In other words, if you *already* know that $\dim V = m$, then any m linearly independent vectors in V automatically span V, and any m vectors that span V are automatically linearly independent.

Why?

- If you had m linearly independent vectors that don't form a basis, then they don't span. Hence you can find another vector in V but not in the span of these m, to get $m + 1$ linearly independent vectors. The span of these has dimension $m + 1$. But a subspace of dimension m can't contain a subspace of larger dimension.
- If you had m vectors that span but don't form a basis, they're linearly dependent. This means you can remove a vector to get $m - 1$ vectors that span V. This means $\dim V < m$.

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix, and let $T : \mathbf{R}^n \to \mathbf{R}^n$ be the linear transformation $T(x) = Ax$. The following statements are equivalent.

1. A is invertible.

2. T is invertible.
3. A is row equivalent to I_n.
4. A has n pivots.
5. $Ax = 0$ has only the trivial solution.
6. The columns of A are linearly independent.
7. T is one-to-one.

8. $Ax = b$ is consistent for all b in $\mathbf{R}^n$.
9. The columns of A span $\mathbf{R}^n$.
10. T is onto.
11. A has a left inverse (there exists B such that $BA = I_n$).
12. A has a right inverse (there exists B such that $AB = I_n$).
13. A^T is invertible.

14. The columns of A form a basis for $\mathbf{R}^n$.

15. Col $A = \mathbf{R}^n$.

16. dim Col $A = n$.

17. rank $A = n$.

18. Nul $A = \{0\}$.

19. dim Nul $A = 0$.

These are equivalent to the previous conditions by the Rank Theorem and the Basis Theorem. For instance, if the columns of A span $\mathbf{R}^n$, then because there are n columns and dim $\mathbf{R}^n = n$, they form a basis. Hence dim Nul $A = 0$.