Math 1553 Worksheet 6

October 7, 2016

1. Find bases for the column space and the null space of

$$A = \begin{pmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{pmatrix}$$

$$\begin{array}{c} \\ \xrightarrow{RREF} \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

- linearly independant columns of A correspond to columns with pivots

The column space is $\mathbb{R}^2$; is spanned by $\begin{pmatrix} 1 \\ 3 \\ -1 \\ 2 \end{pmatrix}$; $\begin{pmatrix} 4 \\ -2 \\ 0 \\ 3 \end{pmatrix}$

- null space is all vectors that map to $\underline{0}$ $\quad \mathbb{R}^5 \to \mathbb{R}^4$

$Ax = 0$
all solutions of x

$$\begin{aligned} x_1 + x_3 + 2x_4 + x_5 &= 0 \\ x_2 + x_3 + x_4 + 2x_5 &= 0 \\ x_3 = x_3 & \text{ (free)} \\ x_4 = x_4 & \text{ (free)} \\ x_5 = x_5 & \text{ (free)} \end{aligned} \quad \longrightarrow \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \overset{V_1}{\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}} x_3 + \overset{V_2}{\begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}} x_4 + \overset{V_3}{\begin{pmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}} X_5$$

$\boxed{\text{Span}\{v_1, v_2, v_3\} = \text{nullspace}(A)}$

2. Consider the following vectors in $\mathbf{R}^3$:

$$b_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \qquad b_2 = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} \qquad u = \begin{pmatrix} 1 \\ 10 \\ 7 \end{pmatrix}$$

Let $V = \text{Span}\{b_1, b_2\}$.

a) Explain why $\mathcal{B} = \{b_1, b_2\}$ is a basis for V.

1) $\{b_1, b_2\}$ is a basis since we are told it spans V <u>And</u>

2) $\{b_1, b_2\}$ are linearly independant $\longrightarrow$ $\begin{pmatrix} 2 & 1 \\ 2 & 4 \\ 2 & 3 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ pivot in every col so cols of A are lin indep.

b) Determine if u is in V. If it is, find $[u]_{\mathcal{B}}$, the $\mathcal{B}$-coordinate vector of u.

$\vec{u}$ is in V if you can write $\vec{u}$ as a linear combination of the vectors that form a basis of V $(\{b_1, b_2\})$

$$\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} a + \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} b \overset{?}{=} \begin{pmatrix} 1 \\ 10 \\ 7 \end{pmatrix} \to \left(\begin{array}{cc|c} 2 & 1 & 1 \\ 2 & 4 & 10 \\ 2 & 3 & 7 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right) \text{ so } \begin{aligned} a &= -1 \\ b &= 3 \end{aligned}$$

so $\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}(-1) + \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}(3) = \begin{pmatrix} 1 \\ 10 \\ 7 \end{pmatrix}$ so $[U]_B = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

c) Find a vector b_3 such that $\{b_1, b_2, b_3\}$ is a basis of $\mathbf{R}^3$.

$b_3 = \begin{pmatrix} 1 \\ 10 \\ 8 \end{pmatrix}$ Check w/ <u>row reduction</u>

3. Answer "yes" if the statement is always true, "no" if it is always false, and "maybe" otherwise.

Null A

$\dim(\text{free var}) = 98$

YES **a)** If A is a 3×100 matrix of rank 2, then $\dim \mathrm{Nul}\, A = 98$.

3×100 matrix of rank 2 looks like this. $\begin{bmatrix} 1 & 0 & x & x & \cdots & x \\ 0 & 1 & x & x & \cdots & x \\ 0 & 0 & 0 & \cdots & & \end{bmatrix}$ 98

No **b)** If A is an $n \times n$ matrix and $\mathrm{Col}\, A = \mathbf{R}^n$, then $Ax = 0$ has a nontrivial solution.

if cols of A span $\mathbb{R}^n$, then there is a pivot in every row. Since A is $n \times n$, there is also a pivot in every column, so all column vectors of A are linearly independant ∴ therefore can't have non-trivial solution

Maybe **c)** If A is an $m \times n$ matrix and $Ax = 0$ has a nontrivial solution, then the columns of A form a basis for $\mathbf{R}^m$.

free variables exist

true if $\left(\begin{array}{cccc|c} 1 & 0 & 0 & x & 0 \\ 0 & 1 & 0 & x & 0 \\ 0 & 0 & 1 & x & 0 \end{array}\right)$ but not true if $\left(\begin{array}{cccc|c} 1 & 0 & x & x & 0 \\ 0 & 1 & x & x & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$ ← just arbitrary examples.

No **d)** The empty set is a subspace of $\mathbf{R}^m$.

The empty set contains no elements so the very first check for $\vec{0}$ in the set fails. However, $\{\vec{0}\}$ set containing only $\vec{0}$ is a subspace of every space.

4. Which of the following are subspaces of $\mathbf{R}^4$? Why or why not?

Is a Subspace **a)** $V = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; x + y = 0 \text{ and } z + w = 0 \right\}$

Check $\vec{0}$

$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in V$? $\overset{x}{0} + \overset{y}{0} \overset{\checkmark}{=} 0$, $\overset{z}{0} - \overset{w}{0} \overset{\checkmark}{=} 0$

Check if $\vec{v} \in V$ then $c\vec{v} \in V$

(ex) $V = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$ st $v_1 - v_2 = 0$, $v_3 - v_4 = 0$

$cV = \begin{pmatrix} cv_1 \\ cv_2 \\ cv_3 \\ cv_4 \end{pmatrix} \rightarrow$ $cv_1 - cv_2 = c(v_1 - v_2) \overset{\checkmark}{=} 0$, $cv_3 - v_4 = c(v_3 - v_4) \overset{\checkmark}{=} 0$

Check if $\vec{u}, \vec{v} \in V$ then $\vec{u} + \vec{v} \in V$

$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$ $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$ $u_1 - u_2 = 0$, $u_3 - u_4 = 0$, $v_1 - v_2 = 0$, $v_3 - v_4 = 0$

$U + V = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \\ u_4 + v_4 \end{pmatrix} \rightarrow$ $(u_1 + v_1) - (u_2 + v_2) = (u_1 - u_2) + (v_1 - v_2) \overset{\checkmark}{=} 0$

$(u_3 + v_3) - (u_4 + v_4) = (u_3 - u_4) + (v_3 - v_4) \overset{\checkmark}{=} 0$

Not a Subspace **b)** $W = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; xy - zw = 0 \right\}$

Check $\vec{0}$

$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in V$? $\overset{x\,y}{(0)(0)} - \overset{z\,w}{(0)(0)} \overset{\checkmark}{=} 0$

Check $cv \in V$ if $\vec{v} \in V$

$\vec{V} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$ st $v_1 v_2 - v_3 v_4 = 0$

$c\vec{v} = \begin{pmatrix} cv_1 \\ cv_2 \\ cv_3 \\ cv_4 \end{pmatrix}$ $(cv_1)(cv_2) - (cv_3)(cv_4) = c^2(v_1 v_2 - v_3 v_4) \overset{\checkmark}{=} 0$

Check if $\vec{u}, \vec{v} \in V$ then $\vec{u} + \vec{v} \in V$

$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$ st $u_1 u_2 - u_3 u_4 = 0$ & $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$ st $v_1 v_2 - v_3 v_4 = 0$.

$\vec{u} + \vec{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \\ u_4 + v_4 \end{pmatrix} \rightarrow$ $(u_1 + v_1)(u_2 + v_2) - (u_3 + v_3)(u_4 + v_4) =$

$(u_1 u_2 + u_2 v_1 + u_1 v_2 + v_1 v_2) - (u_3 u_4 + u_3 v_4 + u_4 v_3 + v_3 v_4)$

$= (u_1 u_2 - u_3 u_4) + (v_1 v_2 - v_3 v_4) + (u_2 v_1 + u_1 v_2 - u_3 v_4 - u_4 v_3)$

$= u_2 v_1 + u_1 v_2 - u_3 v_4 - u_4 v_3$

↑ not always 0

(ex) where it fails.

$u = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ $v = \begin{pmatrix} 2 \\ 4 \\ 1 \\ 8 \end{pmatrix}$

$u + v = \begin{pmatrix} 3 \\ 5 \\ 2 \\ 9 \end{pmatrix} \rightarrow (3)(5) - (2)(9) \neq 0$